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O(3,3)-like Symmetries of Coupled Harmonic Oscillators

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Abstract

In classical mechanics, the system of two coupled harmonic oscillators is shown to possess the symmetry of the Lorentz group $O(3,3)$ applicable to a six-dimensional space consisting of three space-like and three time-like coordinates, or $SL(4,r)$ in the four-dimensional phase space consisting of two position and two momentum variables. In quantum mechanics, the symmetry is reduced to that of $O(3,2)$ or $Sp(4)$, which is a subgroup of $O(3,3)$ or $SL(4,r)$ respectively. It is shown that among the six $Sp(4)$-like subgroups, only one possesses the symmetry which can be translated into the group of unitary transformations in quantum mechanics. In quantum mechanics, there is the lower bound in the size of phase space for each mode determined by the uncertainty principle while there are no restriction on the phase-space size in classical mechanics. This is the reason why the symmetry is smaller in quantum mechanics.

1 Introduction

For two coupled harmonic oscillators, there is a tendency to believe that the problem is completely and thoroughly understood at the level of Goldstein's textbook on classical mechanics [1] and that no further studies are necessary. We start this paper with the following prejudices.

(a) The group $O(3,3)$ or $SL(4,r)$ is only of mathematical interest, and does not appear to possess any relevance to the physical world, although the fifteen Dirac matrices in the Majorana representation constitute the generators of $O(3,3)$ [2, 3].

(b) The transition from classical to quantum mechanics of this oscillator system is trivial once the problem is brought to a diagonal form with the appropriate normal modes.

(c) The diagonalization requires only a rotation, and no other transformations are necessary.

It is known but not widely known that the diagonalization requires squeeze transformations in addition to the rotation [4, 5, 6]. This means that the diagonalization process in classical mechanics is a symplectic transformation. It is also known that symplectic transformations perform linear canonical transformations in classical mechanics which can be translated into unitary transformations in quantum mechanics [4, 5, 7].

In this paper, we are interested in the size of phase space. In classical mechanics, the size can grow or shrink. We shall show first in this paper that, in classical mechanics, the symmetry group for the two-coupled harmonic oscillators becomes $SL(4,r)$ which is locally isomorphic to $O(3,3)$. This group is not symplectic but has a number of symplectic subgroups which are locally isomorphic to $O(3,2)$ or $O(2,3)$. This conclusion is quite different from our earlier contention that the $O(3,2)$ will solve all the problems for the two-mode oscillator system [8].

Let us translate the above mathematics into the language of physics. In quantum mechanics, the size of phase space is allowed to grow resulting in an increased entropy, but it cannot shrink beyond the limit imposed by the uncertainty principle [7]. If we apply this principle to each of the normal modes, only the $O(3,2)$ symmetry is carried into quantum mechanics.

It is known also that not all the transformations in quantum mechanics are unitary, and the expansion of phase space has its well-defined place in the quantum world [9, 10, 11, 12]. The problem is the shrinking of phase space. In the system of two coupled oscillators, the phase space expansion in one mode means the shrinking phase space in the other mode. While we do not provide a complete solution to the problem, we can give a precise statement of the issue in this paper.

The coupled oscillator problem is covered in freshman physics. It stays with us in many different forms because it provides the mathematical basis for many soluble models in physics, including the Lee model in quantum field theory [13, 14], the Bogoliubov transformation in superconductivity [15, 16, 17], relativistic models of elementary particles [7, 18], and squeezed states of light [2, 19, 20, 21]. In this paper, in addition to the above-mentioned symmetry reduction in quantum mechanics, we shall show that the symmetry consideration of the two coupled oscillators leads to the fifteen Dirac matrices.

In Sec. 2, we construct transformation matrices for the coupled oscillator problem in classical mechanics. In Sec. 3, we show that the $Sp(4)$ symmetry is not enough for full understanding of two coupled oscillators in classical mechanics, and that the group $SL(4,r)$ is needed. It is pointed out that $SL(4,r)$ transformations are not always canonical, and we need non-canonical transformations to deal with the classical oscillator problem. In Sec. 4, we study in detail the local isomorphism between the group $SL(4,r)$ and $O(3,3)$, and use this isomorphism to construct the $Sp(4)$ subgroups of $SL(4,r)$. It is like constructing the $O(3,2)$-like subgroups of $O(3,3)$. In Sec. 5, it is noted that there are three $O(3,2)$-like and three $O(2,3)$-like subgroups in $O(3,3)$. There are therefore six $Sp(4)$ subgroups in $SL(4,r)$. It is then shown that one of them is canonical for a given phase-space coordinate system, and the remaining five are not.

2 Construction of the $Sp(4)$ Symmetry Group from Two Coupled Oscillators

Let us consider a system of two coupled harmonic oscillators. The Hamiltonian for this system is

$$H = \frac{1}{2}\left\{\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + A'x_1^2 + B'x_2^2 + C'x_1x_2\right\}. \tag{1}$$

where

$$A' > 0, \qquad B' > 0, \qquad 4A'B' - C'^2 > 0. \tag{2}$$

By making scale changes of x_1 and x_2 to $(m_1/m_2)^{1/4}x_1$ and $(m_2/m_1)^{1/4}x_2$ respectively, it is possible to make a canonical transformation of the above Hamiltonian to the form [7, 22]

$$H = \frac{1}{2m}\left\{p_1^2 + p_2^2\right\} + \frac{1}{2}\left\{Ax_1^2 + Bx_2^2 + Cx_1x_2\right\}, \tag{3}$$

with $m = (m_1m_2)^{1/2}$. We can decouple this Hamiltonian by making the coordinate transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{4}$$

Under this rotation, the kinetic energy portion of the Hamiltonian in Eq.(3) remains invariant. Thus we can achieve the decoupling by diagonalizing the potential energy. Indeed, the system becomes diagonal if the angle α becomes

$$\tan\alpha = \frac{C}{B - A}. \tag{5}$$

This diagonalization procedure is well known.

As we did in Ref. [8], we introduce the new parameters K and η defined as

$$K = \sqrt{AB - C^2/4}, \qquad \exp(-2\eta) = \frac{A + B + \sqrt{(A-B)^2 + C^2}}{\sqrt{4AB - C^2}}, \tag{6}$$

in addition to the rotation angle α. In terms of this new set of variables, A, B and C take the form

$$\begin{aligned} A &= K\left(e^{2\eta}\cos^2\frac{\alpha}{2} + e^{-2\eta}\sin^2\frac{\alpha}{2}\right), \\ B &= K\left(e^{2\eta}\sin^2\frac{\alpha}{2} + e^{-2\eta}\cos^2\frac{\alpha}{2}\right), \\ C &= K\left(e^{-2\eta} - e^{2\eta}\right)\sin\alpha. \end{aligned} \tag{7}$$

the Hamiltonian can be written as

$$H = \frac{1}{2m}\left\{q_1^2 + q_2^2\right\} + \frac{K}{2}\left\{e^{2\eta}y_1^2 + e^{-2\eta}y_2^2\right\}, \tag{8}$$

where y_1 and y_2 are defined in Eq.(4), and

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} . \tag{9}$$

This form will be our starting point. The above rotation together with that of Eq.(4) is generated by S_3.

If we measure the coordinate variable in units of $(mK)^{1/4}$, and use $(mK)^{-1/4}$ for the momentum variables, the Hamiltonian takes the form

$$H = \frac{1}{2} e^{\eta} \left(e^{-\eta} q_1^2 + e^{\eta} y_1^2 \right) + \frac{1}{2} e^{-\eta} \left(e^{\eta} q_2^2 + e^{-\eta} y_2^2 \right) , \tag{10}$$

where the Hamiltonian is measured in units of $\omega = \sqrt{K/m}$. If $\eta = 0$, the system becomes decoupled, and the Hamiltonian becomes

$$H = \frac{1}{2} \left(p_1^2 + x_1^2 \right) + \frac{1}{2} \left(p_2^2 + x_2^2 \right) . \tag{11}$$

In this paper, we are interested in the transformation of this decoupled Hamiltonian into the most general form given in Eq.(10).

It is important to note that the Hamiltonian of Eq.(11) cannot be obtained from Eq.(10) by a canonical transformation. For this reason, the Hamiltonian of the form

$$H' = \frac{1}{2} \left(e^{-\eta} q_1^2 + e^{\eta} y_1^2 \right) + \frac{1}{2} \left(e^{\eta} q_2^2 + e^{-\eta} y_2^2 \right) \tag{12}$$

may play an important role in our discussion. This Hamiltonian can be transformed into the decoupled form of Eq.(11) through a canonical transformation. We will eventually have to face the problem of transforming the above form to H of Eq.(10), and we shall do this in Sec. 3.

In this section, we are interested in transformations which will bring the uncoupled Hamiltonian of Eq.(11) to H'. For the two uncoupled oscillators, we can start with the coordinate system:

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, p_1, x_2, p_2) . \tag{13}$$

This coordinate system is different from the traditional coordinate system where the coordinate variables are ordered as (x_1, x_2, p_1, p_2). This unconventional coordinate system does not change the physics or mathematics of the problem, but is convenient for studying the uncoupled system as well as expanding and shrinking phase spaces.

Since the two oscillators are independent, it is possible to perform linear canonical transformations on each coordinate separately. The canonical transformation in the first coordinate system is generated by

$$A_1 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C_1 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{14}$$

These generators satisfy the Lie algebra:

$$[A_1, B_1] = iC_1, \qquad [B_1, C_1] = -iA_1, \qquad [C_1, A_1] = iB_1. \tag{15}$$

It is also well known that this set of commutation relations is identical to that for the $(2+1)$-dimensional Lorentz group. Linear canonical transformations on the second coordinate are generated by

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad C_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{16}$$

These generators also satisfy the Lie algebra of Eq.(15). We are interested here in constructing the symmetry group for the coupled oscillators by soldering two $Sp(2)$ groups generated by A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

It will be more convenient to use the linear combinations:

$$\begin{aligned} A_+ &= A_1 + A_2, \qquad B_+ = B_1 + B_2, \qquad C_+ = C_1 + C_2, \\ A_- &= A_1 - A_2, \qquad B_- = B_1 - B_2, \qquad C_- = C_1 - C_2, \end{aligned} \tag{17}$$

These matrices take the form

$$A_+ = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B_+ = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad C_+ = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

$$A_- = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad B_- = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad C_- = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{18}$$

The sets (A_+, B_+, C_+) and (A_+, B_-, C_-) satisfy the Lie algebra of Eq.(15). The same is true for (A_-, B_+, C_-) and (A_-, B_-, C_+).

Next, let us couple the oscillators through a rotation generated by

$$A_0 = \frac{i}{2}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{19}$$

In view of the fact that the first two coordinate variables are for the phase space of the first oscillator, and the third and fourth are for the second oscillator, this matrix generates parallel rotations in the (x_1, x_2) and (p_1, p_2) coordinates. As the coordinates (x_1, x_2) are coupled through a two-by-two matrix, the coordinate (p_1, p_2) are coupled through the same two-by-two matrix.

Then, A_0 commutes with A_+, B_+, C_+, and the following commutation relations generate new operators A_3, B_3 and C_3:

$$[A_0, A_-] = iA_3, \qquad [A_0, B_-] = iB_3, \qquad [A_0, C_-] = iC_3, \tag{20}$$

where

$$A_3 = \frac{1}{2}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad B_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad C_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{21}$$

In this section, we started with the generators of the symmetry groups for two independent oscillators. They are A_1, B_1, C_1 and A_2, B_2, C_2. We then introduced A_0 which generates coupling of two oscillators. This processes produced three additional generators A_3, B_3, C_3. It is remarkable that C_3, B_3 and A_+ form the set of generators for another $Sp(2)$ group. They satisfy the commutation relations

$$[B_3, C_3] = -iA_+, \qquad [C_3, A_+] = iB_3, \qquad [A_+, B_3] = iC_3. \tag{22}$$

The same can be said about the sets A_+, B_1, C_1 and A_+, B_2, C_2. These Sp(2)-like groups are associated with the coupling of the two oscillators.

3 Canonical and Non-canonical Transformations in Classical Mechanics

For a dynamical system consisting of two pairs of canonical variables x_1, p_1 and x_2, p_2, we have introduced the coordinate system $(\eta_1, \eta_2, \eta_3, \eta_4)$ defined in Eq.(13).

The transformation of the variables from η_i to ξ_i is canonical if

$$M J \tilde{M} = J, \tag{23}$$

where

$$M_{ij} = \frac{\partial}{\partial \eta_j} \xi_i,$$

and

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{24}$$

This form of the J matrix appears different from the traditional literature, because we are using the new coordinate system. In order to avoid possible confusion and to maintain continuity with our earlier publications, we give in the Appendix the expressions for the J matrix and the ten generators of the $Sp(4)$ group in the traditional coordinate system. There are four rotation generators and six squeeze generators in this group.

In this new coordinate system, the rotation generators take the form

$$L_1 = \frac{-1}{2} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad L_2 = \frac{i}{2} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

$$L_3 = \frac{-1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad S_3 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{25}$$

The squeeze generators become

$$K_1 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \qquad K_2 = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad K_3 = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad Q_2 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad Q_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{26}$$

There are now ten generators. They form the Lie algebra for the $Sp(4)$ group:

$$[L_i, L_j] = i\epsilon_{ijk} L_k, \qquad [L_i, S_3] = 0,$$

$$[L_i, K_j] = i\epsilon_{ijk} K_k, \qquad [L_i, Q_j] = i\epsilon_{ijk} Q_k,$$

$$[K_i, K_j] = [Q_i, Q_j] = -i\epsilon_{ijk} L_k, \qquad [K_i, Q_j] = -i\delta_{ij} S_3,$$

$$[K_i, S_3] = -iQ_i, \qquad [Q_i, S_3] = iK_i. \tag{27}$$

Indeed, these matrices can be identified with the ten matrices introduced in Sec.2 in the following manner.

$$A_+ = S_3, \quad A_- = -L_3, \quad A_3 = -L_1, \quad A_0 = L_2,$$

$$B_+ = K_2, \qquad B_- = -Q_1, \qquad B_3 = Q_3,$$
$$C_+ = Q_2, \qquad C_- = K_1, \qquad C_3 = -K_3. \tag{28}$$

In Sec. 2, we started with the $Sp(2)$ symmetry for each of the oscillator, and introduced the parallel rotation to couple the system. It is interesting to note that this process leads to the $Sp(4)$ symmetry.

We have chosen the non-traditional phase space coordinate system given in Eq.(13) in order to study the coupling more effectively. The A_0 matrix given in Eq.(19) generates the coupling of two phase spaces by rotation. Within this coordinate system, we are interested in relative adjustments of the sizes of the two phase spaces. Indeed, in order to transform the Hamiltonian of Eq.(12) to that of Eq.(10), we have to expand one phase space while contracting the other. For this purpose, we need the generators of the form

$$G_3 = \frac{i}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{29}$$

This matrix generates scale transformations in phase space. The transformation leads to a radial expansion of the phase space of the first coordinate [23] and contracts the phase space of the second coordinate. What is the physical significance of this operation? The expansion of phase space leads to an increase in uncertainty and entropy. Mathematically speaking, the contraction of the second coordinate should cause a decrease in uncertainty and entropy. Can this happen? The answer is clearly No, because it will violate the uncertainty principle. This question will be addressed in future publications.

In the meantime, let us study what happens when the matrix G_3 is introduced into the set of matrices given in Eq.(25) and Eq.(26). It commutes with S_3, L_3, K_1, K_2, Q_1, and Q_2. However, its commutators with the rest of the matrices produce four more generators:

$$[G_3, L_1] = iG_2, \qquad [G_3, L_2] = -iG_1, \qquad [G_3, K_3] = iS_2, \qquad [G_3, Q_3] = -iS_1, \tag{30}$$

with

$$
\begin{aligned}
G_1 &= \frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad G_2 = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \\
S_1 &= \frac{-i}{2}\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad S_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.
\end{aligned} \tag{31}
$$

If we take into account the above five generators in addition to the ten generators of $Sp(4)$, there are fifteen generators. These generators satisfy the following set of commutation relations.

$$
\begin{gathered}
[L_i, L_j] = i\epsilon_{ijk}L_k, \qquad [S_i, S_j] = i\epsilon_{ijk}S_k, \qquad [L_i, S_j] = 0, \\
[L_i, K_j] = i\epsilon_{ijk}K_k, \qquad [L_i, Q_j] = i\epsilon_{ijk}Q_k, \qquad [L_i, G_j] = i\epsilon_{ijk}G_k, \\
[K_i, K_j] = [Q_i, Q_j] = [Q_i, Q_j] = -i\epsilon_{ijk}L_k, \\
[K_i, Q_j] = -i\delta_{ij}S_3, \qquad [Q_i, G_j] = -i\delta_{ij}S_1, \qquad [G_i, K_j] = -i\delta_{ij}S_2. \\
[K_i, S_3] = -iQ_i, \qquad [Q_i, S_3] = iK_i, \qquad [G_i, S_3] = 0, \\
[K_i, S_1] = 0, \qquad [Q_i, S_1] = -iG_i, \qquad [G_i, S_1] = iQ_i, \\
[K_i, S_2] = iG_i, \qquad [Q_i, S_2] = 0, \qquad [G_i, S_2] = -iK_i.
\end{gathered} \tag{32}
$$

Indeed, the ten $Sp(4)$ generators together with the five new generators form the Lie algebra for the group $SL(4,r)$. This group is known to be locally isomorphic to the Lorentz group $O(3,3)$ with three space variables and three time variables. This means that we can study the symmetry of the two coupled oscillators with this high-dimensional Lorentz group. It is also known that the fifteen Dirac matrices in the Majorana form can serve as the generators of $SL(4,r)$. Thus, the coupled oscillator can also serve as a model for the Dirac matrices. Indeed, this higher-dimensional group could lead to a much richer picture of symmetry than is known today. In the meantime, let us study the local isomorphism between $O(3,3)$ and $SL(4,r)$.

4 Local Isomorphism between $O(3,3)$ and SL(4,r)

In Secs. 2 and 3, we constructed the fifteen generators of the group $SL(4,r)$ from the coupled oscillator system. In this section, we write down the generators of the $O(3,3)$ group and confirm that $O(3,3)$ is locally isomorphic to

$SL(4,r)$. One immediate advantage is to use this isomorphism to construct $Sp(4)$-like subgroups of $SL(4,r)$.

In the Lorentz group $O(3,1)$, there are three rotation generators and three boost generators. In $O(3,2)$, there are three rotation generators for the three space-like coordinates, and one rotation generator for the two time-like coordinates. We can make boosts along the three different space-like directions with respect to each time-like variable. There are therefore two sets of three boost generators for this system with two time-like variables. In this manner, we have studied effectively the $Sp(2)$ subgroups of the group $Sp(4)$. It was interesting to note that there are three $O(1,2)$ subgroups with one space-like and two time-like variables. This is translated into the three corresponding $Sp(2)$ subgroups in $Sp(4)$. The algebraic property of these $O(1,2)$ like subgroups is the same as those for $O(2,1)$-like subgroups.

In the present case of $O(3,3)$, there are now three time-like coordinates. For this three-dimensional space, there are three rotation generators, and there are three sets of boost generators. We should not forget the three rotation generators operating in the three-dimensional space-like space. This is one convenient way to classify the six rotation and nine boost generators in the $O(3,3)$ as well as in the $SL(4,r)$ group where the boost operators become squeeze operators.

With these points in mind, we introduce a six-dimensional space with three space-like variables x, y, z and three time-like variables s, t, u. These variables can be ordered as (x, y, z, s, t, u), and transformations can be performed by six-by-six matrices.

Let us start with the $O(3,2)$ subgroup which was discussed in our earlier papers. The transformations operate in the five-dimensional subspace (x, y, z, s, t). We can still write six-by-six matrices for the generators of this group with zero elements on both sixth rows and columns. Then according to Ref. [8], the generators take the form

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
L_3 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{33}
$$

The Lorentz boosts in the subspace of (x, y, z, t) are generated by

$$
K_1 = \begin{pmatrix} 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,
$$

$$
K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{34}
$$

These three boost generators, together with the rotation generators of Eq.(33), form the Lie algebra for the Lorentz group applicable to the four-dimensional Minkowski space of (x, y, z, t). The same is true for the space of (x, y, z, s) with the boost generators:

$$
Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,
$$

$$
Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{35}
$$

The above two Lorentz groups have nine generators. If we attempt to form a closed set of commutation relations, we end up with an additional

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

which will generate rotations in the two-dimensional space of s and t. These ten generators form a closed set of commutations relations.

According to the commutation relations given in Sec. 3, we can construct two more rotation generators:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix}, \tag{37}$$

which, together with S_3, satisfy the Lie algebra for the three-dimensional rotation group. In addition, there are three additional boost generators:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

These generators satisfy the commutation relations given in Sec. 3, and this confirms the local isomorphism between $O(3,3)$ and $SL(4,r)$ which we constructed in this paper from the coupled oscillators.

Let us now go back to the J matrix of Eq.(24). This J matrix is proportional to S_3 which is one of the rotation operators applicable to three time-like coordinates. This means that S_1 and S_2 can also play their respective roles as the J matrix. If S_1 is proportional to the J matrix, the $O(3,2)$ subgroup operates in the five dimensional space of (x, y, z, t, u). If we use S_2, the subspace is (x, y, z, s, u).

Because there are three time-like coordinates, there are three $O(2,3)$ subgroups. In these cases, each one of the rotation generators L_1, L_2, and L_3 can serve as the J matrix. If L_2 is chosen, for instance, the five-dimensional subspace is (x, z, s, t, u). The corresponding $Sp(4)$-like subgroup is discussed in detail in the Appendix.

It has been known for sometime that the fifteen Dirac matrices can serve as the generators of the group locally isomorphic to $O(3,3)$ [2]. In his recent paper [3], Lee showed that the generators of $SL(4,r)$ indeed constitute the Dirac matrices in the Majorana representation. Thus, we have shown in this paper that the system of two coupled oscillators can serve as a mechanical model for the Dirac matrices.

5 Quantum Mechanics of Coupled Oscillators

We had to construct the $O(3,3)$-like symmetry group in order to transform the Hamiltonian of two uncoupled identical oscillators given in Eq.(11) to the Hamiltonian H' of Eq.(12) and then into the Hamiltonian H of Eq.(10). The ground-state wave function for the two uncoupled oscillators takes the form

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2)\right\}. \tag{39}$$

We do not have to explain in this paper how to construct wave functions for excited states. Our problem is how to transform the above wave function into

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(e^{\eta} y_1^2 + e^{-\eta} y_2^2)\right\}, \tag{40}$$

which is the ground-state wave function for the Hamiltonian H' of Eq.(12). In terms of the x_1 and x_2 variables, this wave function can be written as

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\left[e^{\eta}(x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2\right.\right.$$

$$+e^{-\eta}(x_1 \sin\frac{\alpha}{2} + x_2 \cos\frac{\alpha}{2})^2\Big]\Big\} . \tag{41}$$

It has been shown that there exists a unitary transformation which changes the ground-state wave function of Eq.(39) to the above form [24]. In general, unitary transformations are generated by the differential operators [2]:

$$\begin{aligned}
\hat{L}_1 &= \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \qquad \hat{L}_2 = \frac{1}{2i}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right), \\
\hat{L}_3 &= \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \qquad \hat{S}_3 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2 a_2^\dagger\right), \\
\hat{K}_1 &= -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 - a_2^\dagger a_2^\dagger - a_2 a_2\right), \\
\hat{K}_2 &= \frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 + a_2^\dagger a_2^\dagger - a_2 a_2\right), \\
\hat{K}_3 &= \frac{1}{2}\left(a_1^\dagger a_2^\dagger + a_1 a_2\right), \\
\hat{Q}_1 &= -\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 - a_2^\dagger a_2^\dagger + a_2 a_2\right), \\
\hat{Q}_2 &= -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 + a_2^\dagger a_2^\dagger + a_2 a_2\right), \\
\hat{Q}_3 &= \frac{i}{2}\left(a_1^\dagger a_2^\dagger - a_1 a_2\right).
\end{aligned} \tag{42}$$

where $a^\dagger$ and a are the step-up and step-down operators applicable to harmonic oscillator wave functions. These operators satisfy the Lie algebra for the $Sp(4)$ group in Eq.(27), and there is a one-to-one correspondence between the hatted operators in this section and the unhatted operators in Sec. 3.

Next, we are led to the question of whether there is a transformation in quantum mechanics which corresponds to the transformation of H' of Eq.(12) into H of Eq.(10). It was noted in Sec. 3 that this transformation is non-canonical. At the present time, we do not know how to translate non-canonical transformations into the language of the Schrödinger picture of quantum mechanics where only unitary transformations are allowed. We are thus unable add five more hatted operators to the above list of ten generators.

Under these circumstances, the best we can do is to use the same wave function for both H and H' with different expressions for the eigenvalues. For H, the energy eigenvalues are

$$E_{n_1,n_2} = n_1 + n_2 + 1, \tag{43}$$

where n_1 and n_2 are the excitation numbers for the first and second modes respectively. On the other hand, the eigenvalues for H' are

$$E_{n_1,n_2} = e^{\eta}n_1 + e^{-\eta}n_2 + 1. \tag{44}$$

On the other hand, there is a provision for nonunitary transformations in the density matrix formulation of quantum mechanics [9, 25]. One way to deal with this problem for the present case is to use the Wigner phase space picture of quantum mechanics [26].

For two-mode problems, the Wigner function is defined as [7]

$$\begin{aligned} W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int \exp\left\{-2i(p_1 y_1 + p_2 y_2)\right\} \\ \times\, \psi^*(x_1 + y_1, x_2 + y_2)\psi(x_1 - y_1, x_2 - y_2) dy_1 dy_2. \end{aligned} \tag{45}$$

The Wigner function corresponding to the oscillator wave function of Eq.(40) is

$$\begin{aligned} W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\Big\{-e^{\eta}(x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2 \\ -e^{-\eta}(x_1 \sin\frac{\alpha}{2} + x_2 \cos\frac{\alpha}{2})^2 - e^{-\eta}(p_1 \cos\frac{\alpha}{2} - p_2 \sin\frac{\alpha}{2})^2 \\ -e^{\eta}(p_1 \sin\frac{\alpha}{2} + p_2 \cos\frac{\alpha}{2})^2\Big\}. \end{aligned} \tag{46}$$

Indeed, the Wigner function is defined over the four-dimensional phase space of (x_1, p_1, x_2, p_2) just as in the case of classical mechanics. The unitary transformations generated by the operators of Eq.(42) are translated into linear canonical transformations of the Wigner function [24]. The canonical transformations are generated by the differential operators [7]:

$$L_1 = +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_2}\right)\right\},$$

$$
\begin{aligned}
L_2 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) + \left(p_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_1}\right)\right\}, \\
L_3 &= +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_1}\right) - \left(x_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_2}\right)\right\}, \\
S_3 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_2}\right)\right\},
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
K_1 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_1}\right) - \left(x_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_2}\right)\right\}, \\
K_2 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} - p_1\frac{\partial}{\partial p_1}\right) + \left(x_2\frac{\partial}{\partial x_2} - p_2\frac{\partial}{\partial p_2}\right)\right\}, \\
K_3 &= +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_2}\right)\right\}, \\
Q_1 &= +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} - p_1\frac{\partial}{\partial p_1}\right) - \left(x_2\frac{\partial}{\partial x_2} - p_2\frac{\partial}{\partial p_2}\right)\right\}, \\
Q_2 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_2}\right)\right\}, \\
Q_3 &= -\frac{i}{2}\left\{\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) - \left(p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}\right)\right\}.
\end{aligned} \tag{48}
$$

These differential operators are the same as their matrix counterparts given in Sec. 3.

Unlike the case of the Schrödinger picture, it is possible to add five non-canonical generators to the above list. They are

$$
\begin{aligned}
S_1 &= +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) - \left(p_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_1}\right)\right\}, \\
S_2 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_2}\right)\right\},
\end{aligned} \tag{49}
$$

as well as three additional squeeze operators:

$$
\begin{aligned}
G_1 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}\right) + \left(p_1\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial p_1}\right)\right\}, \\
G_2 &= \frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_1}\right) - \left(x_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_2}\right)\right\}, \\
G_3 &= -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} + p_1\frac{\partial}{\partial p_1}\right) + \left(x_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_2}\right)\right\}. \qquad (50)
\end{aligned}
$$

These five generators perform well defined operations on the Wigner function from the mathematical point of view. However, the question is whether these additional generators are acceptable in the present form of quantum mechanics.

In order to answer this question, let us note that the uncertainty principle in the phase-space picture of quantum mechanics is stated in terms of the minimum area in phase space for a given pair of conjugate variables. The minimum area is determined by Planck's constant. Thus we are allowed to expand the phase space, but are not allowed to contract it. With this point in mind, let us go back to G_3 of Eq.(29), which generates transformations which simultaneously expand one phase space and contract the other. Thus, the G_3 generator is not acceptable in quantum mechanics even though it generates well-defined mathematical transformations of the Wigner function.

Unlike the matrix generators, the form of differential operators applicable to the four-dimensional phase space is invariant under reordering of the coordinate variables. Of course, there are six different ways to choose ten generators from fifteen to construct the $O(3,2)$-like subgroups. Thus However, only one of them can be accommodated in the present form of quantum mechanics. The rotation generators S_1, S_2 and the squeeze generators G_1, G_2, G_3 cannot generate meaningful transformations in quantum mechanics. The question of whether they are useful in the quantum world remains as an interesting future problem.

In this paper, we started with one of the most elementary physical systems, and noted that it leads to one of the most sophisticated symmetry problems in physics. It is remarkable that the differential operators given in this Section can serve as Dirac matrices. These operators generate geometric transformations in a four-dimensional space. Indeed many attempts have been made in the past to give geometric interpretations of the Dirac matrices, and the present paper is not likely to be the last one.

Generators of Sp(4) in the Traditional Notation

This Appendix has two different purposes. First, in order to maintain continuity, we give the expressions for the generators for canonical transformations used in the traditional literature including our own papers where the phase-space coordinates are (x_1, x_2, p_1, p_2). Second, we point that they do not generate canonical transformations in the new coordinate system we introduced in Sec. 3.

For a dynamical system consisting of two pairs of canonical variables x_1, p_1 and x_2, p_2, we can use the coordinate variables defined as

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, x_2, p_1, p_2)\,. \tag{51}$$

Then the transformation of the variables from η_i to ξ_i is canonical if

$$MJ\tilde{M} = J, \tag{52}$$

where

$$M_{ij} = \frac{\partial}{\partial \eta_j}\xi_i,$$

and

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We used this coordinate system in our earlier papers, and the transformation from the coordinate system of Eq.(13) to this coordinate system is straight-forward. The generators of the $SP(4)$ group are

$$L_1 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad L_2 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

$$L_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \qquad S_3 = \frac{i}{2}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{53}$$

The following six symmetric generators anticommute with J.

$$K_1 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad K_2 = \frac{i}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad K_3 = -\frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

and

$$Q_1 = \frac{i}{2}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad Q_2 = \frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad Q_3 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{54}$$

These generators satisfy the set of commutation relations for $Sp(4)$ given in Sec. 3 :

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \qquad [L_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = [Q_i, Q_j] = -i\epsilon_{ijk}L_k,$$
$$[L_i, S_3] = 0, \qquad [K_i, Q_j] = -i\delta_{ij}S_3,$$
$$[L_i, Q_j] = i\epsilon_{ijk}Q_k, \qquad [K_i, S_3] = -iQ_i, \qquad [Q_i, S_3] = iK_i. \tag{55}$$

These generators indeed satisfy the Lie algebra for the $Sp(4)$ group. However, the above J matrix is different from the J matrix of Eq.(24), and the above set of generators is different from the set given in Eqs. (25) and (26). This means that the above generators form a $Sp(4)$-like subgroup different from the one discussed in Secs. 3 and 4.

In the new coordinate system, the above J matrix is proportional to L_2. The remaining rotation generators are S_1, S_2, S_3, and the boost generators are K_3, Q_3, G_3 and K_1, Q_1, G_1. This $Sp(4)$ subgroup is like one of the three $O(2,3)$-like subgroup of $O(3,3)$. Because the operators G_3 and G_1 are non-canonical, this is not a canonical subgroup.

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