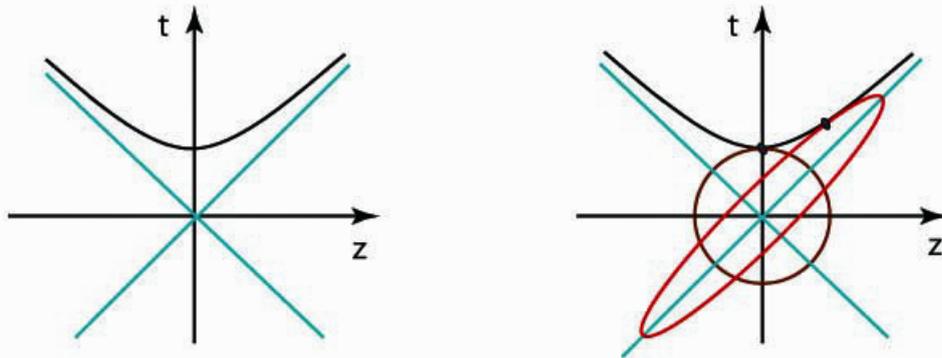


# Physics of the Lorentz Group



To Einstein's hyperbola of 1905,  
we have added a circle and squeezed it.



# Physics of the Lorentz Group

Sibel Başkal

Department of Physics, Middle East Technical University, 06800 Ankara, Turkey  
e-mail: [baskal@newton.physics.metu.edu.tr](mailto:baskal@newton.physics.metu.edu.tr)

Young S. Kim

Center for Fundamental Physics, University of Maryland, College Park, Maryland  
20742, U.S.A.  
e-mail: [yskim@umd.edu](mailto:yskim@umd.edu)

Marilyn E. Noz

Department of Radiology, New York University, New York, NY 10016 U.S.A.  
e-mail: [marilyn.noz@med.nyu.edu](mailto:marilyn.noz@med.nyu.edu)



# Preface

When Newton formulated his law of gravity, he wrote down his formula applicable to two point particles. It took him 20 years to prove that his formula works also for extended objects such as the sun and earth.

When Einstein formulated his special relativity in 1905, he worked out the transformation law for point particles. The question is what happens when those particles have space-time extensions. The hydrogen atom is a case in point. The hydrogen atom is small enough to be regarded as a particle obeying Einstein's law of Lorentz transformations including the energy-momentum relation  $E = \sqrt{p^2 + m^2}$ .

Yet, it is known to have a rich internal space-time structure, rich enough to provide the foundation of quantum mechanics. Indeed, Niels Bohr was interested in why the energy levels of the hydrogen atom are discrete. His interest led to the replacement of the orbit by a standing wave.

Before and after 1927, Einstein and Bohr met occasionally to discuss physics. It is possible that they discussed how the hydrogen atom with an electron orbit or as a standing-wave looks to moving observers. However, there are no written records. If they were not able to see this problem, it is because there were and still are no hydrogen atoms with relativistic speed.

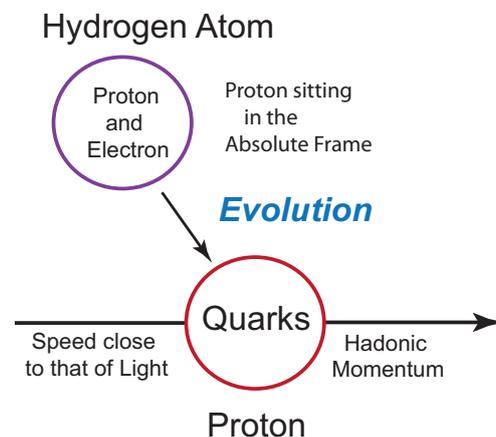


Figure 1: Evolution of the hydrogen atom. The proton and hydrogen atom share the same quantum mechanics of bound states. Unlike the hydrogen atom, the proton can be accelerated, and its speed can become extremely close to that of light.

However, an evolution has taken place in the way we look at the hydrogen atom. These days, there are moving protons. Fortunately, the proton is also a bound state of more fundamental particles called quarks. Since the proton and the hydrogen atom share the

same quantum mechanics, it is possible to study the original Bohr-Einstein problem of moving hydrogen atoms while looking at accelerated protons. This transition is shown in Fig. 1.

In 1971 in an attempt to construct a Lorentz-covariant picture of the quark model, Feynman and his students wrote down a Lorentz-invariant differential equation for the harmonic oscillator potential. This partial differential equation has many different solutions depending on the choice of coordinate systems and boundary conditions.

Earlier, in 1927, 1945, and 1949, Paul A. M. Dirac noted the problem of constructing wave functions which can be Lorentz-boosted. He had to approach this problem mathematically because there were no moving bound states. In 1949, he concluded that the solution to this problem is to construct a suitable representation of the Poincaré group.

Indeed, the purpose of this book is to develop mathematical tools to approach this problem. In 1939, Eugene Wigner published a paper dealing with subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. If the momentum is invariant, these subgroups deal with the internal space-time symmetries. For instance, for a massive particle at rest, the subgroup is  $O(3)$  or the three-dimensional rotation group. Spherical harmonics constitute a representation of the three-dimensional rotation group. Likewise, it is possible to construct a representation of Wigner's little group for massive particles using harmonic oscillator wave functions.

Wigner however did not deal with the problem of what happens when the  $O(3)$  symmetry is Lorentz-boosted. Of particular interest is what happens when the system is boosted to the infinite-momentum frame. On the other hand, Wigner's 1939 paper provides a framework to carry out this Lorentz completion of his little group, and we shall do so in this book. In so doing it is possible to provide the solutions of the problems left unsolved in the papers of Dirac and Feynman.

This Lorentz completion allows us to deal with the Bohr-Einstein question of how the hydrogen atom appears to a moving observer. We can study the same problem using harmonic oscillator wave functions, and study what we observe in high-energy particle physics. If the proton is at rest, it is a bound state just like the hydrogen atom. If the proton moves with a velocity close to that of light, it appears like a collection of Feynman's partons. The Lorentz completion therefore shows that the quark and parton models are two limiting cases of one covariant entity just as the case of  $E = p^2/2m$  and  $E = cp$  are two limiting cases of  $E = \sqrt{p^2 + m^2}$ .

While the group of transformations applicable to the four-dimensional Minkowskian space is represented by four-by-four matrices, it is possible to represent the same group with two-by-two matrices. This allows the study of the property of the group with more transparent matrices. In addition, this group allows the study of mathematical languages for the branch of physics based on two-by-two matrices.

Modern optics is a case in point. Two-by-two matrices serve as the basic mathematical languages for the squeezed state of light, polarization optics, lens optics, and beam transfer matrices commonly called the  $ABCD$  matrices. It is known that those matrices are not rotation matrices because they are the matrices of the Lorentz group. Since Lorentz transformations and modern optics share the same mathematics, it is possible to learn lessons for one subject from the other, as illustrated in Fig. 2.

As for mathematical techniques, what lessons can we learn? We are quite familiar

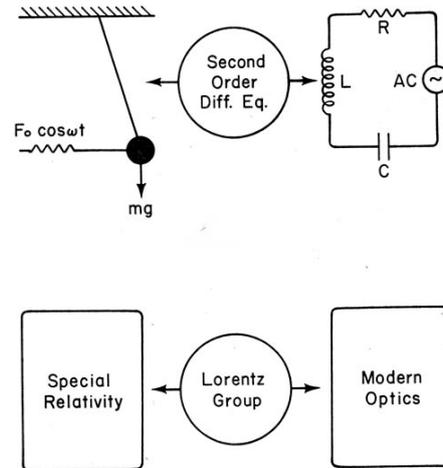


Figure 2: One mathematical language serving two branches of physics. A second-order differential equation can serve as the underlying mathematical language for both the damped harmonic oscillator and the resonance circuit. Likewise, the Lorentz group serves as the underlying language for both special relativity and modern optics.

with the two-by-two matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

which performs rotations in the two-dimensional space of  $x$  and  $y$ . Its transpose becomes its inverse. This matrix is Hermitian.

We can consider another matrix which takes the form

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}.$$

This matrix squeezes the coordinate axes. It expands one axis while contracting the other, in such a way that the area is preserved. This matrix transforms a circle into an ellipse, and its geometry is very familiar to us, but its role in modern physics is not well known largely because it is not a Hermitian matrix. Judicious combinations of the rotation and squeeze matrices lead to a very effective mathematics capable of addressing many different aspects of modern physics.



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# Chapter 1

## Lorentz group and its representations

The Lorentz group starts with a group of four-by-four matrices performing Lorentz transformations on the four-dimensional Minkowski space of  $(t, z, x, y)$ . The transformation leaves invariant the quantity  $(t^2 - z^2 - x^2 - y^2)$ . There are three generators of rotations and three boost generators. Thus, the Lorentz group is a six-parameter group.

It was Einstein who observed that this Lorentz group is applicable also to the four-dimensional energy and momentum space of  $(E, p_z, p_x, p_y)$ . In this way, he was able to derive his Lorentz-covariant energy-momentum relation commonly known as  $E = mc^2$ . This transformation leaves  $(E^2 - p_z^2 - p_x^2 - p_y^2)$  invariant. In other words, the particle mass is a Lorentz invariant quantity.

### 1.1 Generators of the Lorentz Group

Let us start with rotations applicable to the  $(z, x, y)$  coordinates. The four-by-four matrix for this operation is

$$Z(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad (1.1)$$

which can be written as

$$Z(\phi) = \exp(-i\phi J_3), \quad (1.2)$$

with

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \quad (1.3)$$

The matrix  $J_3$  is known as the generator of the rotation around the  $z$  axis. It is not difficult to write the generators of rotations around the  $x$  and  $y$  axes, and they can be

written as  $J_1$  and  $J_2$  respectively, with

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1.4)$$

These three rotation generators satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \quad (1.5)$$

The matrix which performs the Lorentz boost along the  $z$  direction is

$$B(\eta) = \begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.6)$$

with

$$B(\eta) = \exp(-i\eta K_3), \quad (1.7)$$

with the generator

$$K_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1.8)$$

It is then possible to write the matrices for the generators  $K_1$  and  $K_2$ , as

$$K_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \quad (1.9)$$

Then

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \quad \text{and} \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (1.10)$$

There are six generators of the Lorentz group, and they satisfy the three sets of commutation relations given in Eq. (1.5) and Eq. (1.10). It is said that the Lie algebra of the Lorentz group consists of these sets of commutation relations.

These commutation relations are invariant under Hermitian conjugation. While the rotation generator is Hermitian, the boost generators are anti-Hermitian

$$J_i^\dagger = J_i, \quad \text{while} \quad K_i^\dagger = -K_i. \quad (1.11)$$

Thus, it is possible to construct two four-by-four representations of the Lorentz group, one with  $K_i$  and the other with  $-K_i$ . For this purpose we shall use the notation (Berestetskii 1982, Kim and Noz 1986)

$$\dot{K}_i = -K_i. \quad (1.12)$$

Since there are two representations, transformations with  $K_i$  are called the covariant transformations, while those with  $\dot{K}_i$  are called contravariant transformations.

## 1.2 Two-by-two representation of the Lorentz group

It is possible to construct the Lie algebra of the Lorentz group from the three Pauli matrices (Dirac 1945b, Naimark 1954, Kim and Noz 1986, Başkal *et al.* 2014). Let us define

$$J_i = \frac{1}{2}\sigma_i, \quad \text{and} \quad K_i = \frac{i}{2}\sigma_i, \quad (1.13)$$

These two-by-two matrices satisfy the Lie algebra of the Lorentz group given in Eq. (1.5) and Eq. (1.10).

These generators will lead to a two-by-two matrix of the form

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (1.14)$$

with four complex matrix elements, thus eight real parameters. Since its determinant is fixed and is equal to one, there are six independent parameters. This six-parameter group is commonly called  $SL(2, c)$ . Since the Lorentz group has six generators, this two-by-two matrix can serve as a representation of the Lorentz group. It is said in the literature that  $SL(2, c)$  serves as the covering group for the Lorentz group.

For each  $G$  matrix of  $SL(2, c)$ , there exists one four-by-four Lorentz transformation matrix. We can start with the Minkowskian four-vector  $(t, z, x, y)$  written as

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}, \quad (1.15)$$

whose determinant is

$$t^2 - z^2 - x^2 - y^2. \quad (1.16)$$

The correspondence between the two-by-two and four-by-four representations of the Lorentz group along with the generators are given in Table 1.1. These representations can be used for coordinate or momentum transformations, as well as other four-vector quantities such as electromagnetic four-potentials. We can now consider the transformation

$$X' = G X G^\dagger, \quad (1.17)$$

The transformation of Eq. (1.17) can be explicitly written as

$$\begin{pmatrix} t' + z' & x' - iy' \\ x' + iy' & t' - z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \quad (1.18)$$

We can now translate this formula into

$$\begin{pmatrix} t' + z' \\ t' - z' \\ x' - iy' \\ x' + iy' \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha & \gamma^*\beta & \gamma^*\alpha & \alpha^*\beta \\ \beta^*\gamma & \delta^*\delta & \delta^*\gamma & \beta^*\delta \\ \beta^*\alpha & \delta^*\alpha & \beta^*\beta & \delta^*\beta \\ \alpha^*\gamma & \gamma^*\gamma & \alpha^*\delta & \gamma^*\delta \end{pmatrix} \begin{pmatrix} t + z \\ t - z \\ x - iy \\ x + iy \end{pmatrix}. \quad (1.19)$$

This then leads to

$$\begin{pmatrix} t' \\ z' \\ x' \\ y' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix} \begin{pmatrix} t' + z' \\ t' - z' \\ x' - iy' \\ x' + iy' \end{pmatrix}. \quad (1.20)$$

Table 1.1: Two-by-two and four-by-four representations of the Lorentz group.

| Generators  | Two-by-two  | Four-by-four   |
|---|---|--|
| $J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} \exp(i\phi/2) & 0 \\ 0 & \exp(-i\phi/2) \end{pmatrix}$   | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}$                |
| $K_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ | $\begin{pmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{pmatrix}$   | $\begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$             |
| $J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  | $\begin{pmatrix} \cos(\theta/2) & i \sin(\theta/2) \\ i \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$          | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}$        |
| $K_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$  | $\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$      | $\begin{pmatrix} \cosh \lambda & 0 & \sinh \lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \lambda & 0 & \cosh \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |
| $J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$             | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$        |
| $K_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\begin{pmatrix} \cosh(\lambda/2) & -i \sinh(\lambda/2) \\ i \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ | $\begin{pmatrix} \cosh \lambda & 0 & 0 & \sinh \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \lambda & 0 & 0 & \cosh \lambda \end{pmatrix}$ |

It is important to note that the transformation of Eq. (1.17) is not a similarity transformation. In the  $SL(2, c)$  regime, not all the matrices are Hermitian (Başkal *et al.* 2014)

Likewise, the two-by-two matrix for the four-momentum of the particle takes the form

$$P = \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix} \quad (1.21)$$

with  $p_0 = \sqrt{m^2 + p_z^2 + p_x^2 + p_y^2}$ . The transformation of this matrix takes the same form as that for space-time given in Eqs. (1.17) and (1.18). The determinant of this matrix is  $m^2$  and remains invariant under Lorentz transformations. The explicit form of the transformation is

$$\begin{aligned}
P' &= G P G^\dagger = \begin{pmatrix} p'_0 + p'_z & p'_x - ip'_y \\ p'_x + ip'_y & p'_0 - p'_z \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \tag{1.22}
\end{aligned}$$

### 1.3 Representations based on harmonic oscillators

The matrix representations in the previous section are primarily for coordinate transformations. The question then is how can we transform functions. This problem has a stormy history. For plane waves, the form

$$\exp(ip \cdot x) \tag{1.23}$$

is widely used in the literature. Since

$$p \cdot x = Et - p_x x - p_y y - p_z z$$

is a Lorentz-invariant quantity, there are no problems from the mathematical point of view.

However, for standing waves, we have to consider boundary conditions. The issue is then how to transform these conditions. One way to circumvent this difficulty is to study harmonic oscillators with built-in boundary conditions.

Indeed, Dirac (Dirac 1945a, 1963), Yukawa (Yukawa 1953) and Feynman (Feynman *et al.* 1971) struggled with this problem using harmonic oscillator wave functions. Later, it was shown possible to construct the representation of the Poincaré group for relativistic extended particles based on harmonic oscillators (Kim *et al.* 1979, Kim and Noz 1986). This representation serves useful purposes in understanding high-speed hadrons. We shall discuss these problems systematically in Chapters 5 and 6.

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# Chapter 2

## Wigner's little groups for internal space-time symmetries

When Einstein formulated his special relativity, he was interested in point particles, without internal space-time structures. For instance, particles can have intrinsic spins. Massless photons have helicities. The hydrogen atom is a bound state of the electron and proton with a nonzero size. The question is how these particles look to moving observers.

In order to address this question, let us study Wigner's little groups. In 1939, Wigner (Wigner 1939) considered the subgroup of the Lorentz group whose transformations leave the particle momentum invariant. On the other hand, they can transform the internal space-time structure of the particles. Since the particle momentum is fixed and remains invariant, we can assume that the particle momentum is along the  $z$  direction.

This momentum is invariant under rotations around this axis. In addition, these rotations commute with the Lorentz boost along the  $z$  axis. According to the Lie algebra of Eq. (1.10),

$$[J_3, K_3] = 0. \quad (2.1)$$

With these preparations, we can simplify the problem using the Euler coordinate system (Goldstein 1980). Euler formulated his coordinate system in order to understand spinning tops in classical mechanics. In quantum mechanics, we study this problem by constructing representations of the rotation group, such as the spherical harmonics and Pauli matrices. Now the pressing issue is what happens if the system is Lorentz-boosted.

### 2.1 Euler decomposition of Wigner's little group

The Euler angles constitute a convenient parameterization of the three-dimensional rotations (Goldstein 1980). The Euler kinematics consists of two rotations around the  $z$  axis with one rotation around the  $y$  axis between them. These three operations cover also the rotation around the  $x$  axis, thanks to the commutation relation

$$[J_2, J_3] = iJ_1. \quad (2.2)$$

In this way, it is possible to study the essential features of three-dimensional rotations using the two dimensional space of  $z$  and  $x$ . This aspect is well known.

The first question is what happens if we add a Lorentz boost along the  $z$  direction to this traditional procedure (Han *et al.* 1986). Since the rotation around the  $z$  axis is not affected by the boost along the same axis, we are asking what happens to the rotation around the  $y$  axis if it is boosted along the  $z$  direction.

## 2.2 $O(3)$ -like little group for massive particles

If the particle has a positive mass, there is a Lorentz frame in which the particle is at rest, with its four-momentum proportional to

$$P = (1, 0, 0, 0). \quad (2.3)$$

This momentum remains invariant under rotations. Thus, the little group of the massive particle at rest is the three-dimensional rotation group.

The three generators of this little group are  $J_1$ ,  $J_2$ , and  $J_3$ , satisfying the Lie algebra of Eq. (1.5). The dynamical variables associated with these Hermitian operators are known to be particle spins.

The system can be boosted along the  $z$  axis, with the boost matrix  $B(\eta)$  given in Eq. (1.6). If this matrix is applied to the four-momentum of Eq. (2.3), it becomes

$$P' = (\cosh \eta, \sinh \eta, 0, 0). \quad (2.4)$$

The generators become

$$J'_i = B(\eta)J_iB^{-1}(\eta). \quad (2.5)$$

Under this boost operation,  $J_3$  remains invariant, but  $J'_2$  becomes

$$J'_2 = (\cosh \eta)J_2 - (\sinh \eta)K_1. \quad (2.6)$$

As for  $J_1$ , the boost results in

$$J'_1 = (\cosh \eta)J_1 + (\sinh \eta)K_2. \quad (2.7)$$

However, we can obtain the same result by rotating  $J'_2$  by  $-90^\circ$  around the  $z$  axis, thanks to the Euler effect discussed in Sec. 2.1.

Although the generators  $J'_i$  satisfy the same Lie algebra as that for  $J_i$ , they are not the same. Thus, we shall call the operators  $J'_i$  generators of the  $O(3)$ -like little group for the massive particle with a non-zero momentum.

An interesting issue is what happens when  $\eta$  becomes infinite, and  $\sinh \eta = \cosh \eta$ . We shall discuss this problem in Chapter 4.

## 2.3 $E(2)$ -like little group for massless particles

If the particle is massless, its four-momentum is proportional to

$$P = (1, 1, 0, 0). \quad (2.8)$$

This expression is of course invariant under rotations around the  $z$  axis.

In addition, Wigner (Wigner 1939) observed that it is also invariant under the transformation

$$D(\gamma, \phi) = \exp[-i\gamma(N_1 \sin \phi + N_2 \cos \phi)] \quad (2.9)$$

with

$$N_1 = J_1 + K_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 \end{pmatrix}, \quad N_2 = K_1 - J_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & i & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.10)$$

As a consequence,

$$D(\gamma, \phi) = \begin{pmatrix} 1 + \gamma^2/2 & -\gamma^2/2 & \gamma \cos \phi & \gamma \sin \phi \\ \gamma^2/2 & 1 - \gamma^2/2 & \gamma \cos \phi & \gamma \sin \phi \\ \gamma \cos \phi & -\gamma \cos \phi & 1 & 0 \\ \gamma \sin \phi & -\gamma \sin \phi & 0 & 1 \end{pmatrix}. \quad (2.11)$$

Thus the generators of the little group are  $N_1$ ,  $N_2$ , and  $J_3$ . They satisfy the following set of commutation relations.

$$[N_1, N_2] = 0, \quad [N_1, J_3] = iN_2, \quad [N_2, J_3] = -iN_1. \quad (2.12)$$

As Wigner notes, this Lie algebra is the same as that for the two-dimensional Euclidean group, with

$$[P_1, P_2] = 0, \quad [P_1, J_3] = iP_2, \quad [P_2, J_3] = -iP_1, \quad (2.13)$$

where  $P_1$  and  $P_2$  generate translations along the  $x$  and  $y$  directions respectively. They can be written as

$$P_1 = -i\frac{\partial}{\partial x}, \quad P_2 = -i\frac{\partial}{\partial y}, \quad (2.14)$$

while the rotation generator  $J_3$  takes the form

$$J_3 = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \quad (2.15)$$

In addition, Kim and Wigner (Kim and Wigner 1987) considered the following operators,

$$Q_1 = -ix\frac{\partial}{\partial z}, \quad Q_2 = iy\frac{\partial}{\partial z}, \quad (2.16)$$

with  $x^2 + y^2 = \text{constant}$ . They generate translations along the  $z$  direction on the surface of a circular cylinder as described in Fig. 2.1. Then they satisfy the following commutation relations:

$$[Q_1, Q_2] = 0, \quad [Q_1, J_3] = iQ_2, \quad [Q_2, J_3] = -iQ_1. \quad (2.17)$$

We can say that this is the Lie algebra for the ‘‘cylindrical group.’’

Let us consider a photon whose momentum is along the  $z$  direction. It has the four-potential

$$(A_0, A_z, A_x, A_y). \quad (2.18)$$

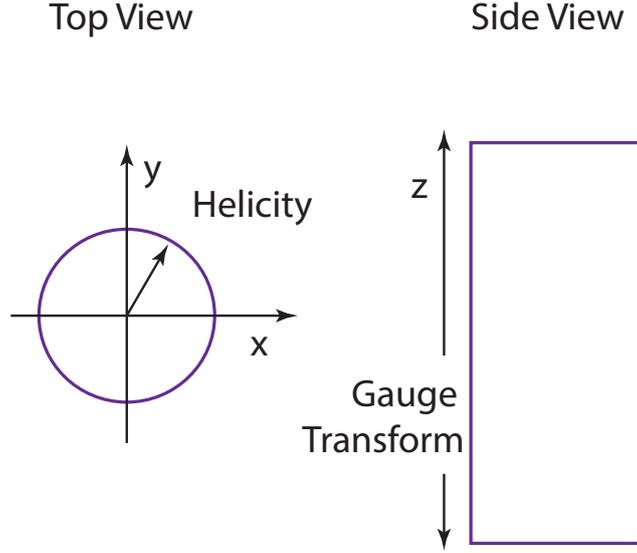


Figure 2.1: Cylindrical picture of the internal space-time structure of photons. The top view of this cylinder is a circle whose rotational degree of freedom corresponds to the helicity of the photon, while the top-down translation corresponds to a gauge transformation (Kim and Wigner 1990).

According to the Lorentz condition,  $A_0 = A_z$ . Thus the four-potential is

$$(A_0, A_0, A_x, A_y). \quad (2.19)$$

If we apply the  $D(\gamma, \phi)$  of Eq. (2.11):

$$\begin{pmatrix} 1 + \gamma^2/2 & -\gamma^2/2 & \gamma \cos \phi & \gamma \sin \phi \\ \gamma^2/2 & 1 - \gamma^2/2 & \gamma \cos \phi & \gamma \sin \phi \\ \gamma \cos \phi & -\gamma \cos \phi & 1 & 0 \\ \gamma \sin \phi & -\gamma \sin \phi & 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_0 \\ A_x \\ A_y \end{pmatrix}, \quad (2.20)$$

the result is

$$\begin{pmatrix} A_0 + \gamma (A_x \cos \phi + A_y \sin \phi) \\ A_0 + \gamma (A_x \cos \phi + A_y \sin \phi) \\ A_x \\ A_y \end{pmatrix}. \quad (2.21)$$

If we boost the four-momentum of Eq. (2.8) along the  $z$  direction, the four-momentum becomes

$$P' = e^\eta(1, 1, 0, 0), \quad (2.22)$$

and  $N_1$  and  $N_2$  become  $e^\eta N_1$  and  $e^\eta N_2$  respectively.  $J_3$  remains invariant.

The little group transformation  $D(\gamma, \phi)$  leaves the transverse components  $A_x$  and  $A_y$  invariant, but provides an addition to  $A_0$ . This is a cylindrical transformation. In the language of physics, it is a gauge transformation. This is summarized in Table 2.1.

## 2.4. O(2,1)-LIKE LITTLE GROUP FOR IMAGINARY-MASS PARTICLES 11

Table 2.1: Covariance of the energy-momentum relation, and covariance of the internal space-time symmetry. Under the Lorentz boost along the  $z$  direction,  $J_3$  remains invariant, and this invariant component of the angular momentum is called the helicity. The transverse component  $J_1$  and  $J_2$  collapse into a gauge transformation. The  $\gamma$  parameter for the massless case has been studied in earlier papers in the four-by-four matrix formulation (Han *et al.* 1982, Kim and Wigner 1990). This table is from (Han *et al.* 1986).

| Massive, Slow       | COVARIANCE            | Massless, Fast                   |
|---------------------|-----------------------|----------------------------------|
| $E = p^2/2m$        | Einstein's $E = mc^2$ | $E = cp$                         |
| $J_3$<br>$J_1, J_2$ | Wigner's Little Group | Helicity<br>Gauge Transformation |

## 2.4 O(2,1)-like little group for imaginary-mass particles

We are now interested in transformations which leave the four-vector of the form

$$P = (0, 1, 0, 0) \quad (2.23)$$

invariant. Then  $P^2 = -1$ , and it is a negative number. We are accustomed to positive values of  $P^2 = (mass)^2$ . This means that the particle mass is imaginary, and it moves faster than light. We are thus talking about a particle we cannot observe in the real world. On the other hand, these particles play a major theoretical role in Feynman diagrams.

We are now interested in transformations which leave the four-vector of Eq. (2.23) invariant. Let us consider the Lorentz boost along the  $y$  direction, with

$$S(\lambda) = \begin{pmatrix} \cosh \lambda & 0 & 0 & \sinh \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \lambda & 0 & 0 & \cosh \lambda \end{pmatrix}, \quad (2.24)$$

which is generated by  $K_2$ . Likewise, it is invariant under the boost along the  $x$  direction. Thus, we can consider the set of commutation relations

$$[J_3, K_1] = iK_2, \quad [J_3, K_2] = -iK_1, \quad [K_1, K_2] = -iJ_3. \quad (2.25)$$

This is a Lie algebra of the Lorentz group applicable to two space and one time dimensions. This group is known in the literature as  $O(2, 1)$ .

If we boost the four-momentum of Eq. (2.23) along the  $z$  direction, it becomes

$$(\sinh \eta, \cosh \eta, 0, 0), \quad (2.26)$$

while  $J_3$  remains invariant.  $K_1$  and  $K_2$  become

$$K'_1 = (\cosh \eta)K_1 - (\sinh \eta)J_2, \quad K'_2 = (\cosh \eta)K_2 + (\sinh \eta)J_1, \quad (2.27)$$

respectively. The generators  $K'_1, K'_2$ , and  $J_3$  satisfy the same Lie algebra as that of Eq. (2.25).

If we interchange  $\sinh \eta$  and  $\cosh \eta$ , these generators become those of Eq. (2.6). If  $\eta$  becomes very large, the four-momentum takes the same form for the massive, massless, and imaginary mass cases. The generators of the little groups also become the same. We shall discuss this issue in Chapter 4.

Even though we are talking about imaginary-mass particles in this section, the mathematics of this little group is applicable to many branches of physics.

Table 2.2: Wigner four-vectors and Wigner matrices applicable to two space-like and one time-like dimensions. Each Wigner four-vector remains invariant under the application of its Wigner matrix.

| Mass           | Wigner Four-vector | Wigner Matrix  |
|----------------|--------------------|--|
| Massive        | $(1, 0, 0, 0)$     | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$                          |
| Massless       | $(1, 1, 0, 0)$     | $\begin{pmatrix} 1 + \gamma^2/2 & -\gamma^2/2 & \gamma & 0 \\ \gamma^2/2 & 1 - \gamma^2/2 & \gamma & 0 \\ \gamma & -\gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |
| Imaginary mass | $(0, 1, 0, 0)$     | $\begin{pmatrix} \cosh \lambda & 0 & \sinh \lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \lambda & 0 & \cosh \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$                   |

## 2.5 Summary

In this Chapter, we discussed the Lorentz group applicable to one time and two space dimensions, namely to the  $(z, x, t)$  coordinates. The rotation around the  $z$  axis will leave the four-momenta listed in this Chapter invariant. It will also extend the representation to the the full  $(z, x, y, t)$  space. With this point in mind, we can list the four-vectors and

matrices which leave their respective four-vectors invariant in Table 2.2. Since Wigner (1939) constructed these four-vectors and four-by-four matrices, we can call them Wigner four-vectors and Wigner matrices. The Lorentz frames where they take these forms can be called Wigner frame.

According to Table 2.1, we are encouraged to look for one covariant expression for all three cases of Table 2.2. We shall discuss this problem in Chapter 4.

The  $E(2)$ -like little group for massless particles was formulated by 1939 (Wigner 1939), but its physical interpretation was not completely settled until 1990 (Kim and Wigner 1990). This problem had a stormy history of forty one years, and a comprehensive list of earlier papers on this subject was given by Kim and Wigner in their 1987 paper (Kim and Wigner 1987). The debate is still continuing, and there are recent papers on this subject (Scaria and Chakraborty 2002, Lindner *et al.* 2003, Caban and Rembielinski 2003).

As for the Lorentz group applicable to the  $(2 + 1)$ -dimensional space, physical applications seldom go beyond transformations applicable in the  $z$  and  $x$  coordinates. This smaller group is called  $O(2, 1)$ . Calculations in high-energy physics involving Lorentz transformations are mostly based on  $O(2, 1)$  (Kim and Noz 1986). In addition, this group serves as one of the basic languages in classical and modern optics (Kim and Noz 1991, Başkal and Kim 2013). We shall discuss these physical applications in later chapters of this book.

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# Chapter 3

## Two-by-two representations of Wigner's little groups

It was noted in Sec. 1.2 that the Lorentz transformation of the four-momentum can be represented by two-by-two matrices. An explicit form for the Lorentz transformation was given in Eq. (1.22) as (Kim and Noz 1986, Başkal *et al.* 2014)

$$P' = G P G^\dagger, \quad (3.1)$$

where the two-by-two form of the  $G$  matrix is given in Eq. (1.14). If the particle moves along the  $z$  direction, the four-momentum matrix becomes

$$P = \begin{pmatrix} E + p & 0 \\ 0 & E - p \end{pmatrix}, \quad (3.2)$$

where  $E$  and  $p$  are the energy and the magnitude of momentum respectively.

Let us use  $W$  as a subset of matrices which leaves the four-momentum invariant, then we can write

$$P = W P W^\dagger. \quad (3.3)$$

These matrices of course constitute Wigner's little group dictating the internal the space-time symmetry of the particle.

If the particle is massive, it can be brought to the system where it is at rest with  $p = 0$ . The four-momentum matrix is proportional to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.4)$$

For the massless particle,  $E = p$ . Thus the four-momentum matrix is proportional to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.5)$$

If the particle mass is imaginary, there is a Lorentz frame where the energy component vanishes. Thus, the Wigner four-vector becomes

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.6)$$

For all three cases, the matrix of the form

$$Z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \quad (3.7)$$

will satisfy the Wigner condition of Eq. (3.3). This matrix corresponds to rotations around the  $z$  axis.

### 3.1 Representations of Wigner's little groups

For a massive particle, since the momentum matrix is proportional to the unit matrix, the  $W$  matrix should be Hermitian, and the little group is the  $SU(2)$  subgroup of the Lorentz group, namely the rotation subgroup. According to Euler, it is sufficient to consider rotations around the  $y$  axis, using this rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (3.8)$$

together with the rotation matrix  $Z(\phi)$ .

If rotated around by  $z$  axis, it becomes

$$Z(\phi)R(\theta)Z^\dagger(\phi) = \begin{pmatrix} \cos(\theta/2) & -e^{i\phi} \sin(\theta/2) \\ e^{-i\phi} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \quad (3.9)$$

However, according the Euler decomposition, it is sufficient to consider only the  $R(\theta)$  with  $\phi = 0$  as the representation of Wigner's  $O(3)$ -like little group for massive particles.

If the particle is massless, the Wigner matrix is necessarily triangular and should take the form

$$T(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \quad (3.10)$$

This matrix has properties that are not too familiar to us. First of all, it cannot be diagonalized. Its inverse and Hermitian conjugate are

$$T^{-1}(\gamma) = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad T^\dagger = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}, \quad (3.11)$$

respectively. Since they are not the same,  $T$  is not a Hermitian matrix.

If we rotate the  $T$  matrix around the  $z$  axis, it becomes

$$Z(\phi)T(\gamma)Z^\dagger(\phi) = \begin{pmatrix} 1 & -e^{i\phi}\gamma \\ 0 & 1 \end{pmatrix}, \quad (3.12)$$

Thus, we shall use the triangular matrix of Eq. (3.10) as the representation of the group.

For a particle with an imaginary mass, we can choose the  $W$  matrix as

$$\exp(-i\lambda K_1) = \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}. \quad (3.13)$$

Table 3.1: The Wigner momentum vectors in the two-by-two matrix representation together with the corresponding transformation matrix. These four-momentum matrices have determinants which are positive, zero, and negative for massive, massless, and imaginary-mass particles, respectively

| Particle mass  | Four-momentum                                   | Transform matrix   |
|----------------|---|--|
| Massive        | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$        |
| Massless       | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  | $\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$   |
| Imaginary mass | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ |

This transformation leaves the four-momentum of Eq. (3.6) invariant. If rotated around the  $z$  axis, it becomes

$$\begin{pmatrix} \cosh(\lambda/2) & e^{i\phi} \sinh(\lambda/2) \\ e^{-i\phi} \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}. \quad (3.14)$$

However, it is sufficient to choose the real matrix of Eq. (3.13) as the  $O(2,1)$ -like little group for imaginary particles.

Table 3.1 summarizes the transformation matrices for Wigner's little groups for massive, massless, and imaginary-mass particles.

## 3.2 Lorentz completion of the little groups

We are now interested in boosting the Wigner four-vectors and the representation matrices along the  $z$  direction. The boosted four-momentum is

$$P' = B(\eta) P B^\dagger(\eta), \quad (3.15)$$

with

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad (3.16)$$

The boosted four-momentum should then take the form

$$\begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix}, \quad (3.17)$$

Table 3.2: Lorentz-boosted Wigner vectors and the Wigner matrices in the two-by-two representation. They take the same form for infinite values of  $\eta$ , if the parameters  $\theta$ ,  $\lambda$ , and  $\gamma$  are made to decrease by  $e^{-\eta}$ .

| Particle mass  | Four-momentum  | $W'$ matrix   |
|----------------|--|---|
| Massive        | $\begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix}$  | $\begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$        |
| Massless       | $\begin{pmatrix} e^\eta & 0 \\ 0 & 0 \end{pmatrix}$          | $\begin{pmatrix} 1 & -e^\eta \gamma \\ 0 & 1 \end{pmatrix}$   |
| Imaginary mass | $\begin{pmatrix} e^\eta & 0 \\ 0 & -e^{-\eta} \end{pmatrix}$ | $\begin{pmatrix} \cosh(\lambda/2) & e^\eta \sinh(\lambda/2) \\ e^{-\eta} \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ |

for the massive particle, and

$$\begin{pmatrix} e^\eta & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} e^\eta & 0 \\ 0 & -e^{-\eta} \end{pmatrix}. \quad (3.18)$$

respectively for massless and imaginary-mass particles.

However, the boosted Wigner matrix becomes

$$W' = B(\eta) W B^{-1}(\eta). \quad (3.19)$$

It should be noted that  $B^\dagger(\eta)$  is not the same as  $B^{-1}(\eta)$ . This is a similarity transformation, unlike the transformation of the four-momentum given in Eq. (3.15).

The boosted  $W$  matrix becomes

$$W' = \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (3.20)$$

for the massive particle. For massless and imaginary particles, they become

$$\begin{pmatrix} 1 & e^\eta \gamma \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \cosh(\lambda/2) & e^\eta \sinh(\lambda/2) \\ e^{-\eta} \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}, \quad (3.21)$$

respectively.

These results are tabulated in Table 3.2. In the limit of large  $\eta$ , all three momentum matrices take the same form. If they are multiplied by  $e^{-\eta}$ , they all become the four-momentum of the massless particle. The boosted Wigner matrices become  $\sin(\theta/2)$ ,  $\gamma$ , and  $\sinh(\lambda/2)$ . This leads us to look for three little groups as three different branches of one little group. We shall discuss this problem more systematically in Chapter 4.

### 3.3 Bargmann and Wigner decompositions

Let us restate the contents of Sec. 3.2. In the case of a massive particle moving with the four-momentum  $P'$  of Eq. (3.17), the representation of the little group is  $W'$  of Eq. (3.20). This is also a Lorentz-boosted form of the  $W$  matrix. Indeed, it can be written as

$$B(\eta)WB(-\eta) = B(\eta) \{B(-\eta) [B(\eta)WB(-\eta)] B(\eta)\} B(-\eta), \quad (3.22)$$

meaning that the system is brought back to the frame in which the particle is at rest, a rotation is made without changing the momentum, and then the system is boosted back to a frame where it gains regains its original momentum. This three-step operation is illustrated on the left side in Fig. 3.1. Since this operation consists of three matrices, we shall call it the Wigner decomposition.

However, this is not the only momentum-preserving transformation. We can start with a momentum along the  $z$  direction, as illustrated in right hand side of Fig. 3.1. We can rotate this momentum around the  $y$  axis, boost along the negative  $x$  direction, and then rotate back to the original momentum along the  $z$  direction. Then this operation can be written as

$$R(\alpha)S(-2\chi)R(\alpha) \quad (3.23)$$

This is a product of one boost matrix sandwiched between two rotation matrices. This form is called the Bargmann decomposition (Bargmann 1947).

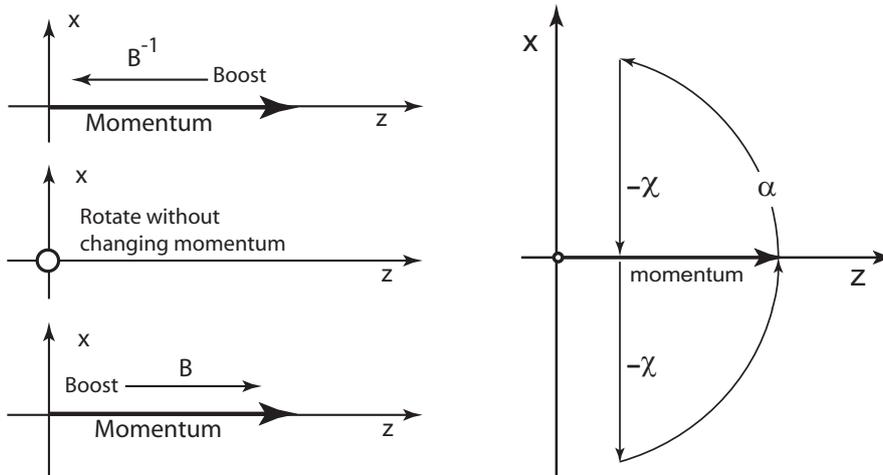


Figure 3.1: Wigner decomposition (left) and Bargmann decomposition (right). These figures illustrate momentum preserving transformations. In the Wigner transformation, a massive particle is brought to its rest frame. It can be rotated while the momentum remains the same. This particle is then boosted back to the frame with its original momentum. In the Bargmann decomposition, the momentum is rotated, boosted, and rotated to its original position (Başkal *et al.* 2014).

The multiplication of these three matrices leads to

$$D(\alpha, \chi) = \begin{pmatrix} (\cos \alpha) \cosh \chi & -\sinh \chi - (\sin \alpha) \cosh \chi \\ -\sinh \chi + (\sin \alpha) \cosh \chi & (\cos \alpha) \cosh \chi \end{pmatrix} \quad (3.24)$$

We can now compare this formula with the momentum preserving  $W'$  matrices given in Table 3.2.

1. If the particle is massive, the off-diagonal elements should have opposite signs,

$$\cos(\theta/2) = (\cos \alpha) \cosh \chi, \quad e^{2\eta} = \frac{(\sin \alpha) \cosh \chi - \sinh \chi}{\sinh \chi + (\sin \alpha) \cosh \chi}, \quad (3.25)$$

with  $(\sin \alpha) \cosh \chi > \sinh \chi$ .

2. If the particle is massless, one of the off-diagonal elements should vanish, and

$$\sinh \chi - (\sin \alpha) \cosh \chi = 0. \quad (3.26)$$

Thus, the diagonal elements become  $(\cos \alpha) \cosh \chi = 1$ , and the non-vanishing off-diagonal element becomes  $2 \sinh \chi = \gamma$ .

3. If the particle mass is imaginary, the off-diagonal elements have the same sign,

$$\cosh(\lambda/2) = (\cosh \chi) \cos \alpha, \quad \text{and} \quad e^{-2\eta} = \frac{\sinh \chi - (\cosh \chi) \sin \alpha}{(\cosh \chi) \sin \alpha + \sinh \chi}. \quad (3.27)$$

It is now clear that the transformation matrices become the same triangular form in the limit of large  $\eta$ .

In the Wigner decomposition, the off-diagonal elements do not change their signs. We are assuming all the angles are positive and smaller than  $90^\circ$ . Thus, if we are interested in making a transition from the massive case to the zero-mass, and then to the imaginary cases, we have to make an excursion to the infinite value of  $\eta$ , and then come back through the appropriate route. This is a singular operation in mathematics.

In the Bargmann decomposition, one of the off-diagonal elements can change its sign depending on the parameters  $\alpha$  and  $\chi$ . Thus, the transition from the massive to the massless and imaginary-mass cases is analytic. These mathematical properties are summarized in Table 3.3.

### 3.4 Conjugate transformations

The most general form of the  $SL(2, c)$  matrix is given in Eq. (1.14), with six independent parameters. In terms of the six generators, this matrix can be written as

$$D = \exp \left\{ -i \sum_{i=1}^3 (\theta_i J_i + \eta_i K_i) \right\}, \quad (3.28)$$

where the  $J_i$  are the generators of rotations and the  $K_i$  are the generators of proper Lorentz boosts. They satisfy the Lie algebra given in Eq. (1.10). This set of commutation relations is invariant under the sign change of the boost generators  $K_i$ . Thus, we can consider the “dot conjugation” defined as

$$\dot{D} = \exp \left\{ -i \sum_{i=1}^3 (\theta_i J_i - \eta_i K_i) \right\}, \quad (3.29)$$

Since  $K_i$  are anti-Hermitian while  $J_i$  are Hermitian, the Hermitian conjugate of  $D$  is

$$D^\dagger = \exp \left\{ -i \sum_{i=1}^3 (-\theta_i J_i + \eta_i K_i) \right\}, \quad (3.30)$$

Table 3.3: Bargmann and Wigner decompositions. Their mathematical properties are compared. In the Bargmann decomposition, one analytic expression covers the massive, massless, and imaginary cases. As was noted in Sec. 3.2, the Wigner decomposition is in the form of a similarity transformation which can serve many useful mathematical purposes.

| Decompositions | Analytic | Similarity Trans. |
|----------------|----------|-------------------|
| Bargmann       | Yes      | No                |
| Wigner         | No       | Yes               |

while the Hermitian conjugate of  $\dot{D}$  is

$$\dot{D}^\dagger = \exp \left\{ -i \sum_{i=1}^3 (-\theta_i J_i - \eta_i K_i) \right\}, \quad (3.31)$$

Since we understand the rotation around the  $z$  axis, we can now restrict the kinematics to the  $zt$  plane, and work with the  $Sp(2)$  symmetry. Then the  $D$  matrices can be considered as Bargmann decompositions. First,  $D$  and  $\dot{D}$  are

$$\begin{aligned} D(\alpha, \chi) &= \begin{pmatrix} (\cos \alpha) \cosh \chi & -\sinh \chi - (\sin \alpha) \cosh \chi \\ -\sinh \chi + (\sin \alpha) \cosh \chi & (\cos \alpha) \cosh \chi \end{pmatrix}, \\ \dot{D}(\alpha, \chi) &= \begin{pmatrix} (\cos \alpha) \cosh \chi & \sinh \chi - (\sin \alpha) \cosh \chi \\ \sinh \chi + (\sin \alpha) \cosh \chi & (\cos \alpha) \cosh \chi \end{pmatrix}. \end{aligned} \quad (3.32)$$

These matrices correspond to the ‘‘D loops’’ given in fig.(a) and fig.(b) of Fig. 3.2 respectively. The ‘‘dot’’ conjugation changes the direction of boosts. The dot conjugation leads to the inversion of the space which is called the parity operation.

We can also consider changing the direction of rotations. This results in using the Hermitian conjugates. We can write the Hermitian conjugate matrices as

$$\begin{aligned} D^\dagger(\alpha, \chi) &= \begin{pmatrix} (\cos \alpha) \cosh \chi & -\sinh \chi + (\sin \alpha) \cosh \chi \\ -\sinh \chi - (\sin \alpha) \cosh \chi & (\cos \alpha) \cosh \chi \end{pmatrix}, \\ \dot{D}^\dagger(\alpha, \chi) &= \begin{pmatrix} (\cos \alpha) \cosh \chi & \sinh \chi + (\sin \alpha) \cosh \chi \\ \sinh \chi - (\sin \alpha) \cosh \chi & (\cos \alpha) \cosh \chi \end{pmatrix}. \end{aligned} \quad (3.33)$$

The exponential expressions from Eq. (3.28) to Eq. (3.31), lead to

$$D^\dagger = \dot{D}^{-1}, \quad \text{and} \quad \dot{D}^\dagger = D^{-1}, \quad (3.34)$$

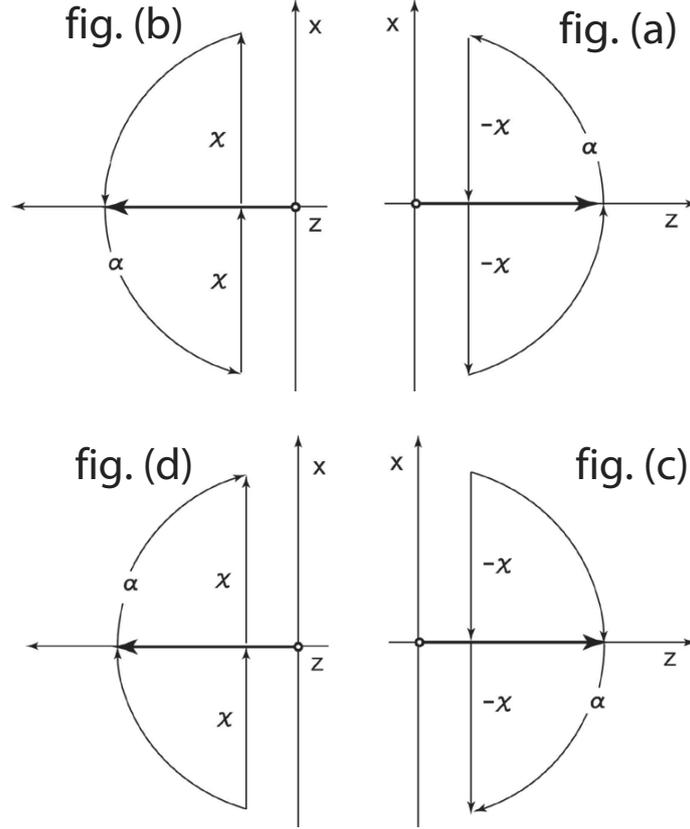


Figure 3.2: Four D-loops resulting from the Bargmann decomposition. Let us go back to Fig. 3.1. If we reverse the direction of the boost, the result is fig.(a). From fig.(a), if we invert the space, we come back to fig.(b). If we reverse the direction of rotation from fig.(a), the result is fig.(c). If both the rotation and space are reversed, the result is the fig.(d) (Başkal *et al.* 2014).

and the Bargmann forms of Eq. (3.32) and Eq. (3.33) are consistent with these relations.

The dot conjugation changes the direction of momentum, and the Hermitian conjugation changes the direction of rotation and the angular momentum. The dot conjugation therefore corresponds to the parity operation, and the Hermitian conjugation to charge conjugation.

### 3.5 Polarization of massless neutrinos

To apply this analysis to spin- $\frac{1}{2}$  particles, we use the group generated by Eq. (1.13)

$$J_i = \frac{1}{2}\sigma_i, \quad \text{and} \quad K_i = \frac{i}{2}\sigma_i. \quad (3.35)$$

These are identical to those for the proper Lorentz group and have the same algebraic properties as the  $SL(2, c)$  group. Additionally, the Lie algebra for these generators is invariant if the sign of the boost operators is changed. In the case of  $SL(2, c)$ , or spin- $\frac{1}{2}$  particles, it is necessary to consider both signs. Also in Chapter 1, we considered that

$SL(2, c)$  consists of non-singular two-by-two matrices which have the form defined in Eq. (1.14)

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (3.36)$$

This matrix is applicable to spinors that have the form:

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.37)$$

for spin-up and spin-down states respectively.

Among the subgroups of  $SL(2, c)$ , there are  $E(2)$ -like little groups which correspond to massless particles. If we consider a massless particle moving along the  $z$  direction, then the little group is generated by  $J_3, N_1$  and  $N_2$ , defined in Eq. (2.10). These  $N$  operators are the generators of gauge transformations in the case of the photon, thus we will refer to them as the gauge transformation in the  $SL(2, c)$  regime (Wigner 1939, Han *et al.* 1982). We shall examine their role with respect to massless particles of spin- $\frac{1}{2}$ .

Since the sign of the generators of this subgroup remains invariant under a sign change in  $K_i$ , these generators remain unambiguous when applied to the space-time coordinate variable and the photon four-vectors. Here we choose the  $J_i$  to be the generators of rotations, but, because of the sign change allowed for  $K_i$  we must consider both  $N_i^{(+)}$  and  $N_i^{(-)}$ , where

$$N_1^{(+)} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad N_2^{(+)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (3.38)$$

The Hermitian conjugates of the above provide  $N_1^{(-)}$  and  $N_2^{(-)}$ . The transformation matrices then can be written as:

$$\begin{aligned} D^{(+)}(\alpha, \beta) &= \exp(-i[\alpha N_1^{(+)} + \beta N_2^{(+)}]) = \begin{pmatrix} 1 & \alpha - i\beta \\ 0 & 1 \end{pmatrix}, \\ D^{(-)}(\alpha, \beta) &= \exp(-i[\alpha N_1^{(-)} + \beta N_2^{(-)}]) = \begin{pmatrix} 1 & 0 \\ -\alpha - i\beta & 1 \end{pmatrix}. \end{aligned} \quad (3.39)$$

Since there are two sets of spinors in  $SL(2, c)$ , the spinors whose boosts are generated by  $K_i = (i/2)\sigma$  will be written as  $u$  and  $v$ , following the usual Pauli notation. For the boosts generated by  $K_i = (-i/2)\sigma$  we will use  $\dot{u}$  and  $\dot{v}$ . These spinors are gauge-invariant in the sense that

$$D^{(+)}(\alpha, \beta)u = u, \quad D^{(-)}(\alpha, \beta)\dot{v} = \dot{v}. \quad (3.40)$$

However, these spinors are gauge-dependent in the sense that

$$D^{(+)}(\alpha, \beta)v = v + (\alpha - i\beta)u, \quad D^{(-)}(\alpha, \beta)\dot{u} = \dot{u} - (\alpha + i\beta)\dot{v} \quad (3.41)$$

The gauge-invariant spinors of Eq. (3.40) appear as polarized neutrinos (Han *et al.* 1982, 1986). In the massless limit,

$$\begin{aligned} D^{(+)}(\alpha, \beta) &= \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \\ D^{(-)}(\alpha, \beta) &= \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}. \end{aligned} \quad (3.42)$$

This is summarized in Table 3.4.

Table 3.4: Hermitian and dot conjugations of the triangular  $T(\gamma)$  matrix.

|                     | Original   | Dot conjugate                                       |
|---------------------|--|---|
| Original            | $\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ |
| Hermitian conjugate | $\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ |

### 3.6 Scalars, four-vectors, and four-tensors

We are quite familiar with the process of constructing three spin-1 states and one spin-0 state from two spinors. Since each spinor has two states, there are four states if combined.

In the Lorentz-covariant world, there are two-more states coming from the dotted representation (Berestetskii 1982, Kim and Noz 1986), as we noted in Sec. 1.1. If four of those two-state spinors are combined, there are 16 states. In this section, we shall construct all sixteen states.

For particles at rest, it is known that the addition of two one-half spins result in spin-zero and spin-one states. Hence, we have two different spinors behaving differently under the Lorentz boost. Around the  $z$  direction, both spinors are transformed by

$$Z(\phi) = \exp(-i\phi J_3) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \quad (3.43)$$

However, they are boosted by

$$B(\eta) = \exp(-i\eta K_3) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix},$$

$$\dot{B}(\eta) = \exp(i\eta K_3) = \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \quad (3.44)$$

applicable to the undotted and dotted spinors respectively. These two matrices commute with each other, and also with the rotation matrix  $Z(\phi)$  of Eq. (3.43). Since  $K_3$  and  $J_3$  commute with each other, we can work with the matrix  $Q(\eta, \phi)$  defined as

$$Q(\eta, \phi) = B(\eta)Z(\phi) = \begin{pmatrix} e^{(\eta-i\phi)/2} & 0 \\ 0 & e^{-(\eta-i\phi)/2} \end{pmatrix},$$

$$\dot{Q}(\eta, \phi) = \dot{B}(\eta)\dot{Z}(\phi) = \begin{pmatrix} e^{-(\eta+i\phi)/2} & 0 \\ 0 & e^{(\eta+i\phi)/2} \end{pmatrix}. \quad (3.45)$$

When this combined matrix is applied to the spinors,

$$Q(\eta, \phi)u = e^{(\eta-i\phi)/2}u, \quad Q(\eta, \phi)v = e^{-(\eta-i\phi)/2}v,$$

$$\dot{Q}(\eta, \phi)\dot{u} = e^{-(\eta+i\phi)/2}\dot{u}, \quad \dot{Q}(\eta, \phi)\dot{v} = e^{(\eta+i\phi)/2}\dot{v}. \quad (3.46)$$

Table 3.5: Sixteen combinations of the  $SL(2, c)$  spinors. In the  $SU(2)$  regime, there are two spinors leading to four bilinear forms. In the  $SL(2, c)$  world, there are two undotted and two dotted spinors. These four-spinors lead to sixteen independent bilinear combinations.

| Spin 1   | Spin 0                                    |
|--|---|
| $uu, \quad \frac{1}{\sqrt{2}}(uv + vu), \quad vv,$                               | $\frac{1}{\sqrt{2}}(uv - vu)$             |
| $i\dot{u}, \quad \frac{1}{\sqrt{2}}(i\dot{v} + \dot{v}i), \quad \dot{v}\dot{v},$ | $\frac{1}{\sqrt{2}}(i\dot{v} - \dot{v}i)$ |
| $u\dot{i}, \quad \frac{1}{\sqrt{2}}(u\dot{v} + \dot{v}i), \quad v\dot{v},$       | $\frac{1}{\sqrt{2}}(u\dot{v} - \dot{v}i)$ |
| $i\dot{u}, \quad \frac{1}{\sqrt{2}}(i\dot{v} + \dot{v}i), \quad \dot{v}\dot{v},$ | $\frac{1}{\sqrt{2}}(i\dot{v} - \dot{v}i)$ |

After the operation of  $Q(\eta, \phi)$  and  $\dot{Q}(\eta, \phi)$

|  |  |
|--|--|
| $e^{-i\phi}e^{\eta}uu, \quad \frac{1}{\sqrt{2}}(uv + vu), \quad e^{i\phi}e^{-\eta}vv,$                               | $\frac{1}{\sqrt{2}}(uv - vu)$                                    |
| $e^{-i\phi}e^{-\eta}i\dot{u}, \quad \frac{1}{\sqrt{2}}(i\dot{v} + \dot{v}i), \quad e^{i\phi}e^{\eta}\dot{v}\dot{v},$ | $\frac{1}{\sqrt{2}}(i\dot{v} - \dot{v}i)$                        |
| $e^{-i\phi}u\dot{i}, \quad \frac{1}{\sqrt{2}}(e^{\eta}u\dot{v} + e^{-\eta}v\dot{v}), \quad e^{i\phi}v\dot{v},$       | $\frac{1}{\sqrt{2}}(e^{\eta}u\dot{v} - e^{-\eta}v\dot{v})$       |
| $e^{-i\phi}i\dot{u}, \quad \frac{1}{\sqrt{2}}(i\dot{v} + \dot{v}i), \quad e^{i\phi}\dot{v}\dot{v},$                  | $\frac{1}{\sqrt{2}}(e^{-\eta}i\dot{v} - e^{\eta}\dot{v}\dot{v})$ |

If the particle is at rest, we can construct the combinations

$$uu, \quad \frac{1}{\sqrt{2}}(uv + vu), \quad vv, \quad (3.47)$$

to obtain the spin-1 state, and

$$\frac{1}{\sqrt{2}}(uv - vu), \quad (3.48)$$

for the spin-zero state. There are four bilinear states. In the  $SL(2, c)$  regime, there are two dotted spinors. If we include both dotted and undotted spinors, there are sixteen independent bilinear combinations. They are given in Table 3.5. This table also gives the effect of the operation of  $Q(\eta, \phi)$ .

Among the bilinear combinations given in Table 3.5, the following two equations are invariant under rotations and also under boosts.

$$S = \frac{1}{\sqrt{2}}(uv - vu), \quad \text{and} \quad \dot{S} = -\frac{1}{\sqrt{2}}(i\dot{v} - \dot{v}i). \quad (3.49)$$

They are thus scalars in the Lorentz-covariant world. Are they the same or different? Let us consider the following combinations

$$S_+ = \frac{1}{\sqrt{2}} (S + \dot{S}), \quad \text{and} \quad S_- = \frac{1}{\sqrt{2}} (S - \dot{S}). \quad (3.50)$$

Under the dot conjugation,  $S_+$  remains invariant, but  $S_-$  changes sign. The boost is performed in the opposite direction and therefore is the operation of space inversion. Thus  $S_+$  is a scalar while  $S_-$  is called a pseudo-scalar.

### 3.6.1 Four-vectors

Let us consider the bilinear products of one dotted and one undotted spinor as  $u\dot{u}$ ,  $u\dot{v}$ ,  $\dot{u}v$ ,  $v\dot{v}$ , and construct the matrix

$$U = \begin{pmatrix} u\dot{v} & v\dot{v} \\ u\dot{u} & v\dot{u} \end{pmatrix}. \quad (3.51)$$

Under the rotation  $Z(\phi)$  and the boost  $B(\eta)$  they become

$$\begin{pmatrix} e^\eta u\dot{v} & e^{-i\phi} v\dot{v} \\ e^{i\phi} u\dot{u} & e^{-\eta} v\dot{u} \end{pmatrix}. \quad (3.52)$$

Indeed, this matrix is consistent with the transformation properties given in Table 3.5, and transforms like the four-vector

$$\begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}. \quad (3.53)$$

This form was given in Eq. (1.15). Under space inversion, this matrix becomes

$$\begin{pmatrix} t - z & -(x - iy) \\ -(x + iy) & t + z \end{pmatrix}. \quad (3.54)$$

This space inversion is known as the parity operation.

The form of Eq. (3.51) for a particle or field with four-components, is given by  $(V_0, V_z, V_x, V_y)$ . The two-by-two form of this four-vector is

$$U = \begin{pmatrix} V_0 + V_z & V_x - iV_y \\ V_x + iV_y & V_0 - V_z \end{pmatrix}. \quad (3.55)$$

If boosted along the  $z$  direction, this matrix becomes

$$\begin{pmatrix} e^\eta (V_0 + V_z) & V_x - iV_y \\ V_x + iV_y & e^{-\eta} (V_0 - V_z) \end{pmatrix}. \quad (3.56)$$

In the mass-zero limit, the four-vector matrix of Eq. (3.56) becomes

$$\begin{pmatrix} 2A_0 & A_x - iA_y \\ A_x + iA_y & 0 \end{pmatrix}, \quad (3.57)$$

with the Lorentz condition  $A_0 = A_z$ . The gauge transformation applicable to the photon four-vector was discussed in detail in Sec. 2.3.

Let us go back to the matrix of Eq. (3.51); we can construct another matrix  $\dot{U}$ . Since the dot conjugation leads to the space inversion,

$$\dot{U} = \begin{pmatrix} \dot{u}v & \dot{v}v \\ \dot{u}u & \dot{v}u \end{pmatrix}. \quad (3.58)$$

Then

$$\begin{aligned} \dot{u}v &\simeq (t - z), & \dot{v}u &\simeq (t + z), \\ \dot{v}v &\simeq -(x - iy), & \dot{u}u &\simeq -(x + iy), \end{aligned} \quad (3.59)$$

where the symbol  $\simeq$  means “transforms like.”

Thus,  $U$  of Eq. (3.51) and  $\dot{U}$  of Eq. (3.58) used up eight of the sixteen bilinear forms. Since there are two bilinear forms in the scalar and pseudo-scalar as given in Eq. (3.50), we have to give interpretations to the six remaining bilinear forms.

### 3.6.2 Second-rank Tensor

In this subsection, we are studying bilinear forms with both spinors dotted and undotted. In Subsec. 3.6.1, each bilinear spinor consisted of one dotted and one undotted spinor. There are also bilinear spinors which are both dotted or both undotted. We are interested in two sets of three quantities satisfying the  $O(3)$  symmetry. They should therefore transform like

$$(x + iy)/\sqrt{2}, \quad (x - iy)/\sqrt{2}, \quad z, \quad (3.60)$$

which are like

$$uu, \quad vv, \quad (uv + vu)/\sqrt{2}, \quad (3.61)$$

respectively in the  $O(3)$  regime. Since the dot conjugation is the parity operation, they are like

$$-\dot{u}\dot{u}, \quad -\dot{v}\dot{v}, \quad -(\dot{u}\dot{v} + \dot{v}\dot{u})/\sqrt{2}. \quad (3.62)$$

In other words,

$$(uu)^\dot{=} = -\dot{u}\dot{u}, \quad \text{and} \quad (vv)^\dot{=} = -\dot{v}\dot{v}. \quad (3.63)$$

We noticed a similar sign change in Eq. (3.59).

In order to construct the  $z$  component in this  $O(3)$  space, let us first consider

$$f_z = \frac{1}{2} [(uv + vu) - (\dot{u}\dot{v} + \dot{v}\dot{u})], \quad g_z = \frac{1}{2i} [(uv + vu) + (\dot{u}\dot{v} + \dot{v}\dot{u})], \quad (3.64)$$

where  $f_z$  and  $g_z$  are respectively symmetric and anti-symmetric under the dot conjugation or the parity operation. These quantities are invariant under the boost along the  $z$  direction. They are also invariant under rotations around this axis, but they are not invariant under boost along or rotations around the  $x$  or  $y$  axis. They are different from the scalars given in Eq. (3.49).

Next, in order to construct the  $x$  and  $y$  components, we start with  $f_\pm$  and  $g_\pm$  as

$$\begin{aligned} f_+ &= \frac{1}{\sqrt{2}} (uu - \dot{u}\dot{u}) & g_+ &= \frac{1}{\sqrt{2}i} (uu + \dot{u}\dot{u}) \\ f_- &= \frac{1}{\sqrt{2}} (vv - \dot{v}\dot{v}) & g_- &= \frac{1}{\sqrt{2}i} (vv + \dot{v}\dot{v}). \end{aligned} \quad (3.65)$$

Then

$$\begin{aligned} f_x &= \frac{1}{\sqrt{2}}(f_+ + f_-) = \frac{1}{2}[(uu - i\dot{u}) + (vv - \dot{v})] \\ f_y &= \frac{1}{\sqrt{2}i}(f_+ - f_-) = \frac{1}{2i}[(uu - i\dot{u}) - (vv - \dot{v})]. \end{aligned} \quad (3.66)$$

and

$$\begin{aligned} g_x &= \frac{1}{\sqrt{2}}(g_+ + g_-) = \frac{1}{2i}[(uu + i\dot{u}) + (vv + \dot{v})] \\ g_y &= \frac{1}{\sqrt{2}i}(g_+ - g_-) = -\frac{1}{2}[(uu + i\dot{u}) - (vv + \dot{v})]. \end{aligned} \quad (3.67)$$

Here  $f_x$  and  $f_y$  are symmetric under dot conjugation, while  $g_x$  and  $g_y$  are anti-symmetric.

Furthermore,  $f_z$ ,  $f_x$ , and  $f_y$  of Eqs. (3.64) and (3.66) transform like a three-dimensional vector. The same can be said for  $g_i$  of Eqs. (3.64) and (3.67). Thus, they can be grouped into the second-rank tensor

$$T = \begin{pmatrix} 0 & -g_z & -g_x & -g_y \\ g_z & 0 & -f_y & f_x \\ g_x & f_y & 0 & -f_z \\ g_y & -f_x & f_z & 0 \end{pmatrix}, \quad (3.68)$$

whose Lorentz-transformation properties are well known. The  $g_i$  components change their signs under space inversion, while the  $f_i$  components remain invariant. They are like the electric and magnetic fields respectively.

If the system is Lorentz-booted,  $f_i$  and  $g_i$  can be computed from Table 3.5. We are now interested in the symmetry of photons by taking the massless limit. According to the procedure developed in Sec. 3.2, we can keep only the terms which become larger for larger values of  $\eta$ . Thus,

$$\begin{aligned} f_x &\rightarrow \frac{1}{2}(uu - \dot{v}), & f_y &\rightarrow \frac{1}{2i}(uu + \dot{v}), \\ g_x &\rightarrow \frac{1}{2i}(uu + \dot{v}), & g_y &\rightarrow -\frac{1}{2}(uu - \dot{v}), \end{aligned} \quad (3.69)$$

in the massless limit.

Then the tensor of Eq. (3.68) becomes

$$F = \begin{pmatrix} 0 & 0 & -E_x & -E_y \\ 0 & 0 & -B_y & B_x \\ E_x & B_y & 0 & 0 \\ E_y & -B_x & 0 & 0 \end{pmatrix}, \quad (3.70)$$

with

$$\begin{aligned} B_x &\simeq \frac{1}{2}(uu - \dot{v}), & B_y &\simeq \frac{1}{2i}(uu + \dot{v}), \\ E_x &= \frac{1}{2i}(uu + \dot{v}), & E_y &= -\frac{1}{2}(uu - \dot{v}). \end{aligned} \quad (3.71)$$

The electric and magnetic field components are perpendicular to each other. Furthermore,

$$E_x = B_y, \quad E_y = -B_x. \quad (3.72)$$

In order to address symmetry of photons, let us go back to Eq. (3.65). In the massless limit,

$$B_+ \simeq E_+ \simeq uu, \quad B_- \simeq E_- \simeq i\dot{v} \quad (3.73)$$

The gauge transformations applicable to  $u$  and  $\dot{v}$  are the two-by-two matrices

$$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}. \quad (3.74)$$

respectively as noted in Sec. 2.3 and Sec. 3.5. Both  $u$  and  $\dot{v}$  are invariant under gauge transformations, while  $\dot{u}$  and  $v$  are not.

The  $B_+$  and  $E_+$  are for the photon spin along the  $z$  direction, while  $B_-$  and  $E_-$  are for the opposite direction. Weinberg (Weinberg 1964) constructed gauge-invariant state vectors for massless particles starting from Wigner's 1939 paper (Wigner 1939). The bilinear spinors  $uu$  and  $i\dot{v}$  correspond to Weinberg's state vectors.

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# Chapter 4

## One little group with three branches

We have noted that  $J_2$  and  $K_1$  can serve as the starters of the representation of Wigner's little groups in their respective Wigner frames. They are for the massive and imaginary-mass particles respectively. It was noted also that, when Lorentz boosted, they become

$$J'_2 = (\cosh \eta)J_2 - (\sinh \eta)K_1, \quad K'_1 = -(\sinh \eta)J_2 + (\cosh \eta)K_1. \quad (4.1)$$

These two equations can be combined into one formula, and we are led to consider the transformation matrix

$$D(x, y) = \exp[-i(yJ_2 - xK_1)] \quad (4.2)$$

as a function of the parameters  $x$  and  $y$ . We shall study this form of the Wigner matrix in detail.

During this study, there will be a problem of handling singularities that are not too familiar to us. We shall use the classical damped harmonic oscillator to study this singularity problem in detail, and conclude that this is not an analytic but can be called a “tangential” continuity.

It was noted in Sec. 2.3 that a cylinder can be used for describing internal space-time symmetry for photons. Later in this section, we shall study how this cylindrical symmetry arises when the system of a massive particle is boosted along the  $z$  direction.

### 4.1 One expression with three branches

Let us write the  $D$  matrix of Eq. (4.2) as (Başkal and Kim 2010, 2013, Başkal *et al.* 2014)

$$D(x, y) = \exp \left\{ \begin{pmatrix} 0 & -(x+y) \\ -x+y & 0 \end{pmatrix} \right\}. \quad (4.3)$$

1. If  $y > x$ , we write

$$x + y = e^\eta \sqrt{y^2 - x^2}, \quad x - y = e^{-\eta} \sqrt{y^2 - x^2}, \quad (4.4)$$

with

$$e^\eta = \sqrt{\frac{x+y}{|y-x|}}, \quad (4.5)$$

and  $D(x, y)$  becomes

$$\exp \left\{ \sqrt{y^2 - x^2} \begin{pmatrix} 0 & -e^\eta \\ e^{-\eta} & 0 \end{pmatrix} \right\} = \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (4.6)$$

with  $\cos(\theta/2) = \cos(\sqrt{y^2 - x^2})$ .

2. If  $x = y$ , this expression becomes

$$D(x, y) = \exp \left\{ \begin{pmatrix} 0 & -2x \\ 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 1 & -2x \\ 0 & 1 \end{pmatrix}. \quad (4.7)$$

This form is for the little group for massless particles, as shown in  $T(\gamma)$  of Eq. (3.10).

3. If  $y < x$ , we can write

$$x - y = e^{-\eta} \sqrt{x^2 - y^2}, \quad x + y = e^\eta \sqrt{x^2 - y^2}. \quad (4.8)$$

Then the  $D$  matrix becomes

$$\exp \left\{ \sqrt{x^2 - y^2} \begin{pmatrix} 0 & -e^\eta \\ -e^{-\eta} & 0 \end{pmatrix} \right\} = \begin{pmatrix} \cosh(\lambda/2) & -e^\eta \sinh(\lambda/2) \\ -e^{-\eta} \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}, \quad (4.9)$$

with

$$\cosh(\lambda/2) = \cosh(\sqrt{x^2 - y^2}). \quad (4.10)$$

This expression is the same as that for the Lorentz-boosted  $W$  matrix given in Eq. (3.21) for imaginary-mass particles, and  $e^\eta$  is given in Eq. (4.5).

Indeed, it is possible to derive three different forms of the  $W'$  matrix. The matrix

$$\begin{pmatrix} 0 & -(x + y) \\ -(x - y) & 0 \end{pmatrix} \quad (4.11)$$

is analytic in the  $x$  and  $y$  variables. However, this  $D$  matrix has three distinct branches. In order to understand the nature of this let us look at what happens when  $x - y$  is a small number

$$\epsilon = x - y. \quad (4.12)$$

We can then write the  $D$  matrix as

$$D(x, \epsilon) = \exp \left\{ \begin{pmatrix} 0 & 2x \\ \epsilon & 0 \end{pmatrix} \right\}. \quad (4.13)$$

If  $\epsilon$  is positive, the Taylor expansion leads to

$$D = \begin{pmatrix} \cosh(\sqrt{2x\epsilon}) & -\left[ \sqrt{2x/\epsilon} \right] \sinh \sqrt{2x\epsilon} \\ -\left[ \sqrt{\epsilon/2x} \right] \sinh(\sqrt{2x\epsilon}) & \cosh(\sqrt{2x\epsilon}) \end{pmatrix}. \quad (4.14)$$

If  $\epsilon$  becomes zero, this expression becomes

$$\begin{pmatrix} 1 & -2x \\ 0 & 1 \end{pmatrix} \quad (4.15)$$

If  $\epsilon$  becomes negative,

$$\sqrt{2x\epsilon} = i\sqrt{-2x\epsilon}, \quad \sqrt{\epsilon/2x} = i\sqrt{-2x/\epsilon}, \quad \sqrt{2x/\epsilon} = -i\sqrt{-2x/\epsilon}, \quad (4.16)$$

if we take  $\sqrt{-1} = i$ . Thus,  $D$  becomes

$$D = \begin{pmatrix} \cos(\sqrt{-2x\epsilon}) & -[\sqrt{-2x/\epsilon}] \sin(\sqrt{-2x\epsilon}) \\ [\sqrt{-\epsilon/2x}] \sin(\sqrt{-2x\epsilon}) & \cos(\sqrt{-2x\epsilon}) \end{pmatrix}. \quad (4.17)$$

The result remains the same if we take  $\sqrt{-1} = -i$ .

This type of singularity is not common in literature. Let us study this point further from a physical example familiar to us.

## 4.2 Classical damped oscillators

Let us start with the second-order differential equation

$$\frac{d^2y}{dt^2} + 2\mu\frac{dy}{dt} + \omega^2y = 0, \quad (4.18)$$

for a classical damped harmonic oscillator. If we introduce the function  $\psi(t)$  as

$$\psi(t) = e^{-\mu t}y(t) \quad (4.19)$$

then  $\psi(t)$  satisfies the simplified differential equation

$$\frac{d^2\psi(t)}{dt^2} + (\omega^2 - \mu^2)\psi(t) = 0. \quad (4.20)$$

This second-order differential equation has two independent solutions. Let us call them  $\psi_1$  and  $\psi_2$ . They satisfy the first-order differential equations

$$\frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & -(\omega + \mu) \\ (\omega - \mu) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (4.21)$$

This coupled equation leads to the second order equation Eq(4.20) for  $\psi_1(t)$  and  $\psi_2(t)$ . The physical solution is an appropriate linear combination of these two wave functions.

The solution of this first order differential equation is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \exp \left\{ \begin{pmatrix} 0 & -(\omega + \mu)t \\ (\omega - \mu)t & 0 \end{pmatrix} \right\} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (4.22)$$

where  $C_1 = \psi_1(0)$  and  $C_2 = \psi_2(0)$ . We can then obtain the solutions by following the procedure developed in Sec. 4.1.

1. If  $\omega > \mu$ , the solution becomes

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega't) & -e^{-\eta} \sin(\omega't) \\ e^{-\eta} \sin(\omega't) & \cos(\omega't) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (4.23)$$

with

$$\omega' = \sqrt{\omega^2 - \mu^2}, \quad \text{and} \quad e^{-\eta} = \sqrt{\frac{\omega + \mu}{|\omega - \mu|}}. \quad (4.24)$$

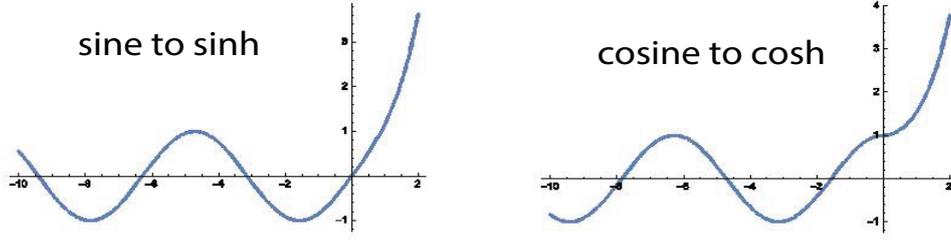


Figure 4.1: Transitions from sine to sinh, and from cosine to cosh. They are continuous transitions. Their first derivatives are also continuous, but the second derivatives are not. Thus, they are not analytically but only tangentially continuous (Başkal and Kim 2014).

2. If  $\omega = \mu$ , the solution becomes

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 1 & -2\omega t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (4.25)$$

3. If  $\mu > \omega$ , the solution matrix becomes

$$\begin{pmatrix} \cosh(\mu't) & e^\eta \sinh(\mu't) \\ e^{-\eta} \sinh(\mu't) & \cosh(\mu't) \end{pmatrix}, \quad (4.26)$$

with  $e^\eta$  given in Eq. (4.24), and

$$\mu' = \sqrt{\mu^2 - \omega^2}. \quad (4.27)$$

Let us now turn to the main issue of what happened when  $\mu$  is close to  $\omega$ . If  $\omega$  is sufficiently close to  $\mu$ , we can let

$$\mu - \omega = 2\mu\epsilon, \quad \text{and} \quad \mu + \omega = 2\omega. \quad (4.28)$$

If  $\omega$  is greater than  $\mu$ , then  $\epsilon$  defined in Eq. (4.28) becomes negative. The solution matrix becomes

$$\begin{pmatrix} 1 - (-\epsilon/2)(2\omega t)^2 & -2\omega t \\ -(-\epsilon)(2\omega t) & 1 - (-\epsilon/2)(2\omega t)^2 \end{pmatrix}, \quad (4.29)$$

which can be written as

$$\begin{pmatrix} 1 - (1/2) [(2\omega\sqrt{-\epsilon}) t]^2 & -2\omega t \\ -\sqrt{-\epsilon} [(2\omega\sqrt{-\epsilon}) t] & 1 - (1/2) [(2\omega\sqrt{-\epsilon}) t]^2 \end{pmatrix}. \quad (4.30)$$

If  $\epsilon$  is positive, Eq. (4.26) can be written as

$$\begin{pmatrix} 1 + (1/2) [(2\omega\sqrt{\epsilon}) t]^2 & -2\omega t \\ -\sqrt{\epsilon} [(2\omega\sqrt{\epsilon}) t] & 1 + (1/2) [(2\omega\sqrt{\epsilon}) t]^2 \end{pmatrix}. \quad (4.31)$$

The transition from Eq. (4.30) to Eq. (4.31) is continuous as they become identical when  $\epsilon = 0$ . As  $\epsilon$  changes its sign, the diagonal elements of above matrices tell us how

Table 4.1: Damped Oscillators and space-time symmetries. They are based on the same set of two-by-two matrices.

|                |                   |                          |
|----------------|-------------------|--------------------------|
| Trace          | Damped Oscillator | Particle Symmetry        |
| Smaller than 2 | Oscillation Mode  | Massive Particles        |
| Equal to 2     | Transition Mode   | Massless Particles       |
| Larger than 2  | Damping Mode      | Imaginary-mass Particles |

$\cos(\omega't)$  becomes  $\cosh(\mu't)$ . The upper-right element remains as  $\sin(\omega't)$  during this transitional process. The lower left element becomes  $\sinh(\mu't)$ . This non-analytic continuity is illustrated in Fig. 4.1.

During this continuation process, the function remains the same. So does its first derivative, but the second derivative does not. Thus, the two functions share the same tangential line. It is indeed a *tangential continuity*. The continuity from one little group to another was discussed in Sec. 4.1. This mathematical similarity is summarized in Table 4.1.

### 4.3 Little groups in the light-cone coordinate system

Speaking of tangential continuity, it was Inönü and Wigner (Inönü and Wigner 1953) who first considered that the  $E(2)$  (two-dimensional Euclidean) group can be constructed as a contraction of the  $O(3)$  group, by considering a flat plane tangential to a sphere. This is very easy to visualize. A football field is clearly a flat two-dimensional surface, but it is also part of the surface of the spherical earth. Thus, the  $E(2)$  group can be constructed as a contraction of the  $O(3)$  group, by considering a flat plane tangential to a sphere. Hence,  $E(2)$  can be constructed from  $O(3)$  in the large-radius limit. For convenience, we can choose a plane tangent to the north pole.

For the same sphere, we can consider also a cylinder tangent to the equatorial belt. We shall consider that the four-by-four representation of the Lorentz group can produce both the  $E(2)$  and the cylindrical symmetries. This aspect is illustrated in fig.(a) of Fig. 4.2.

We noted in Sec. 2.3 that the little groups for massive and massless particles are isomorphic to  $O(3)$  and  $E(2)$  respectively. It is not difficult to construct the  $O(3)$ -like geometry of the little group for a massive particle at rest (Wigner 1939). Let us start with a massive particle at rest. The little group is generated by  $J_1, J_2,$  and  $J_3,$  whose four-by-four forms are given in Sec. 1.1.

For the massless particle, the little group is generated by

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \quad \text{and} \quad J_3. \tag{4.32}$$

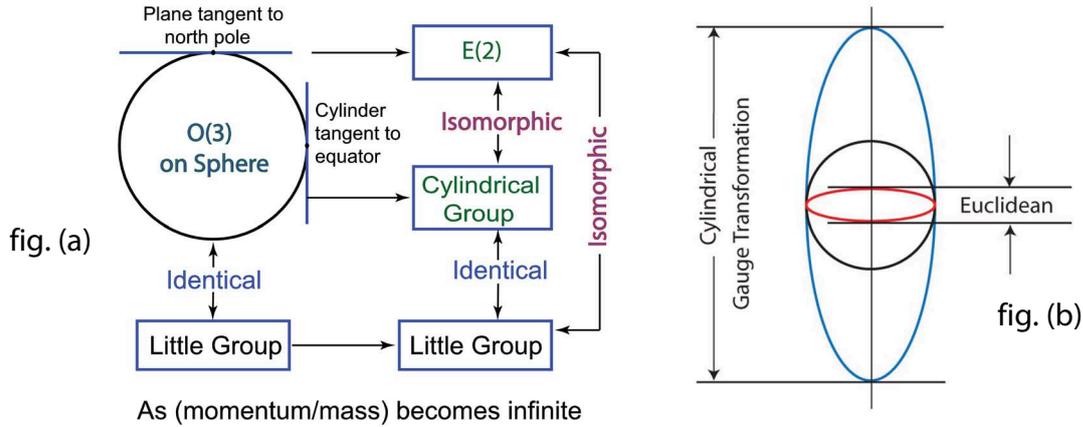


Figure 4.2: Contractions of the three-dimensional rotation group. This group can be illustrated by a sphere. This group can become the two-dimensional Euclidean group on a plane tangent at the north pole as illustrated in fig.(a). It was later noted that there is a cylinder tangential to this sphere, and the up and down translations on this cylinder correspond to the gauge transformation for photons (Kim and Wigner 1987). As illustrated in fig.(b), the four-dimensional representation of the Lorentz group contains both elongation and contraction of the  $z$  axis, as the system is boosted along this direction. The elongation and contraction become the cylindrical and Euclidean groups, respectively (Kim and Wigner 1987, 1990).

These generators satisfy the closed set of commutation relations which constitute the Lie algebra of  $E(2)$  or the two-dimensional Euclidean group.

In this section, we approach the same problem using Dirac's light-cone coordinate system (Dirac 1949), where the variables  $u$  and  $v$  are defined as

$$u = \frac{t+z}{\sqrt{2}}, \quad v = \frac{t-z}{\sqrt{2}}, \quad (4.33)$$

and we have to work with the four-vector  $(u, v, x, y)$ . The major advantage of the light-cone variables is that the Lorentz boost is diagonal. Under the boost, the  $u$  and  $v$  variables become

$$e^\eta u = e^\eta \left( \frac{t+z}{\sqrt{2}} \right), \quad e^{-\eta} v = e^{-\eta} \left( \frac{t-z}{\sqrt{2}} \right), \quad (4.34)$$

while the boost matrix takes the form

$$e^{-i\eta K_3} = \begin{pmatrix} e^\eta & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.35)$$

In the light-cone coordinate system, the generators take the form

$$\begin{aligned}
 J_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & K_3 &= \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 J_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ -i & -i & 0 & 0 \end{pmatrix}, & K_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 J_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & -i & 0 \\ i & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 \end{pmatrix}. \tag{4.36}
 \end{aligned}$$

(4.37)

If a massive particle is at rest, its little group is generated by  $J_1, J_2$ , in addition to  $J_3$ . For a massless particle moving along the  $z$  direction, the little group is generated by  $N_1, N_2$  and  $J_3$ , where

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \tag{4.38}$$

which can be written as

$$N_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \tag{4.39}$$

For convenience, we have dropped the normalization factor of  $\sqrt{2}$ .

These matrices satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \tag{4.40}$$

The transformation generated by the  $N_1$  and  $N_2$  matrices is

$$D(\gamma, \phi) = \exp \{-i\gamma [(\cos \phi)N_1 + (\sin \phi)N_2]\}. \tag{4.41}$$

Since  $N_1^3 = N_2^3 = 0$ , the Taylor expansion of this exponential form truncates, and it becomes

$$\begin{pmatrix} 1 & \gamma^2/2 & \gamma \cos \phi & \gamma \sin \phi \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma \cos \phi & 1 & 0 \\ 0 & -\gamma \sin \phi & 0 & 1 \end{pmatrix}. \tag{4.42}$$

In the light-cone coordinate system, the four components of the vector particle takes the form

$$(A_+, A_-, A_x, A_y) \exp \{i(p_+v + p_+u)\}, \tag{4.43}$$

with

$$A_+ = \frac{A_0 + A_z}{\sqrt{2}}, \quad A_- = \frac{A_0 - A_z}{\sqrt{2}}. \quad (4.44)$$

Thus, for massless photons, the Lorentz condition leads to  $A_0 = A_z$  or  $A_- = 0$ . When the  $D$  matrix is applied to this photon four-vector,

$$\begin{pmatrix} 1 & \gamma^2/2 & \gamma \cos \phi & \gamma \sin \phi \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma \cos \phi & 1 & 0 \\ 0 & -\gamma \sin \phi & 0 & 1 \end{pmatrix} \begin{pmatrix} A_+ \\ 0 \\ A_x \\ A_y \end{pmatrix} = \begin{pmatrix} A_+ + \gamma(A_x \cos \phi + A_y \sin \phi) \\ 0 \\ A_x \\ A_y \end{pmatrix}, \quad (4.45)$$

the transverse components  $A_x$  and  $A_y$  are not affected. The transformation changes only the  $A_0$  and  $A_z$  components. This is a gauge transformation.

This cylindrical transformation has been discussed in Sec. 2.3. The light-cone coordinate system gives a more transparent mathematics. Indeed, this coordinate system is the natural language for the photon internal symmetry.

## 4.4 Lorentz completion in the light-cone coordinate system

As was noted in Sec. 4.3, the matrix for the Lorentz boost is diagonal in the light-cone coordinate system. Let us start with the four-vector for a massive particle

$$V = (V_+, V_-, V_x, V_y). \quad (4.46)$$

If the particle is at rest, this should represent the three-dimensional symmetry, with three independent coordinate variables. This means  $V_+ = V_-$  in the light-cone coordinate system. If we boost the system, this four-vector becomes

$$(e^\eta V_+, e^{-\eta} V_-, V_x, V_y). \quad (4.47)$$

For large values of  $\eta$ , the  $e^{-\eta} V_-$  component vanishes. In the language of photons, it is called the Lorentz condition or  $A_0 = A_z$ . The interesting problem is what happens between those two limits.

To study this problem we can start with two spheres, both with the unit horizontal radius. One of them has the vertical radius  $e^\eta$  and the other with  $e^{-\eta}$ . Since

$$e^\eta = \left[ \frac{\sqrt{p^2 + m^2} + p}{\sqrt{p^2 + m^2} - p} \right]^{1/2}, \quad (4.48)$$

this expression is one when the particle is at rest with  $p = 0$ . In the limit of large values of  $p$ , this number is infinite, while  $e^{-\eta}$  becomes zero.

As  $p$  increases, both go through elliptic deformations as illustrated in fig.(b) of Fig. 4.2. In the large- $\eta$  limit, one of them becomes a cylinder, and the other a flat surface. This picture is equivalent to the tangential plane and cylinder described in fig.(a) of Fig. 4.2.

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## Chapter 5

# Lorentz-covariant harmonic oscillators

Einstein and Bohr met occasionally, before and after 1927, to discuss physics. Einstein was interested in how things look to moving observers, while Bohr was interested in why the energy levels of the hydrogen atom are discrete. Then they must have talked about how the electron orbit of the hydrogen atom looks to a moving observer. There does not seem to be written records to indicate how they sketched the orbits.

However, it is not uncommon to see in the literature the description of the Lorentz deformation as described in Fig. 5.1. This figure became outdated in 1927. The electron orbit is now a standing wave. Thus, the question is how the standing wave appears when the it is boosted along a given direction.

As is indicated in Fig. 5.1, the longitudinal component is affected while the transverse components remain unchanged. With this point in mind, we shall study harmonic oscillators. Since the wave equation for the three-dimensional oscillator is separable in the Cartesian coordinate system, it is sufficient to study the effect of the Lorentz boost only for the longitudinal component of the wave function.

Paul A. M. Dirac spent many years trying to construct a quantum mechanics consistent with special relativity. In his three papers published in 1927, 1945, and 1949 (Dirac 1927, 1945, 1949), Dirac essentially presented all essential ideas on how to construct harmonic oscillator wave functions that can be Lorentz transformed. However, Dirac's most serious problem was the lack of physical phenomena to which his ideas were applicable. The physical example did not exist until Gell-Mann published his paper on the quark model in 1964 (Gell-Mann 1964), where the proton was described as a bound state of three quarks. The proton can be accelerated to speeds very close to that of light.

Dirac (Dirac 1949) said that the problem of Lorentz covariant quantum mechanics is the same as that of constructing a representation of the Poincaré group. Thus, he was telling us to construct the representation based on harmonic oscillators. Since the standing wave is for a bound state, its symmetry is that of Wigner's little group which dictates the internal space-time symmetry (Wigner 1939).

Indeed, the most serious problem in Dirac's paper is that he did not specify the standing wave as a bound state. This problem was clarified later in the paper of Feynman, Kislinger, and Ravndal (Feynman *et al.* 1971). They start with a system of two quarks bound together by a harmonic oscillator force, and write down a Lorentz-invariant dif-

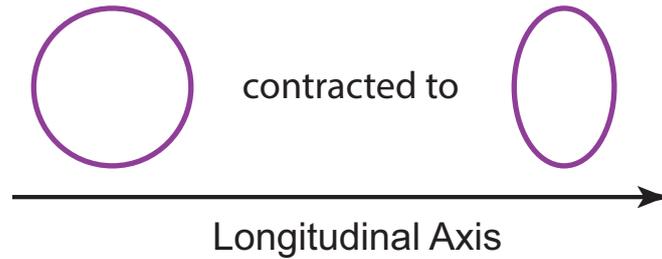


Figure 5.1: Classical picture of Lorentz contraction of the electron orbit in the hydrogen atom. It is expected that the longitudinal component becomes contracted while the transverse components are not affected. In his the first edition of his book published in 1987, 60 years after 1927, John S. Bell included this picture of the orbit viewed by a moving observer (Bell 2004). While talking about quantum mechanics in his book, Bell overlooked the fact that the electron orbit in the hydrogen atom had been replaced by a standing wave in 1927. The question then is how standing waves look to moving observers.

ferential equation. This differential equation has many different forms of solutions. It is possible to construct a set of solutions which constitute a representation of Wigner’s little group. We shall thus start with the Lorentz-invariant differential equation of Feynman *et al.*

## 5.1 Dirac’s plan to construct Lorentz-covariant quantum mechanics

The year 1927 is known for Heisenberg’s uncertainty relation. In the same year, based on the line width and lifetime of excited energy levels, Paul A. M. Dirac formulated his  $c$ -number time energy uncertainty relation (Dirac 1927).

During World War II, Dirac was looking into the possibility of constructing representations of the Lorentz group using harmonic oscillator wave functions (Dirac 1945). The Lorentz group is the language of special relativity, and the present form of quantum mechanics starts with harmonic oscillators. Therefore, he was interested in making quantum mechanics Lorentz covariant by constructing representations of the Lorentz group using harmonic oscillators.

In his 1945 paper (Dirac 1945), Dirac considers the Gaussian form

$$\exp \left\{ -\frac{1}{2} (x^2 + y^2 + z^2 + t^2) \right\}. \quad (5.1)$$

We note that this Gaussian form is in the  $(x, y, z, t)$  coordinate variables. Thus, if we consider a Lorentz boost along the  $z$  direction, we can drop the  $x$  and  $y$  variables, and write the above equation as

$$\exp \left\{ -\frac{1}{2} (z^2 + t^2) \right\}. \quad (5.2)$$

This is a strange expression for those who believe in Lorentz invariance where  $(z^2 - t^2)$  is an invariant quantity.

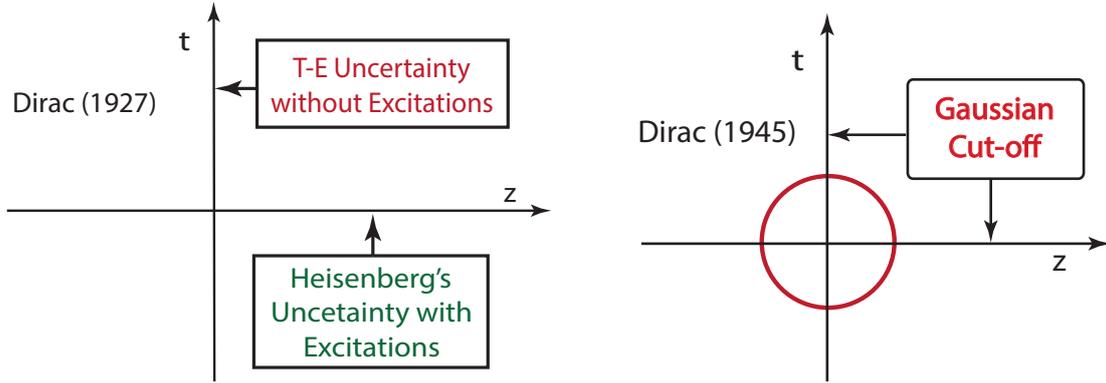


Figure 5.2: Space-time picture of quantum mechanics. In 1927, Dirac noted that there is an uncertainty in the time and energy variables, but there are no excitations along the time axis. He called it the “c-number” time-energy relation. In 1945, Dirac attempted to construct a representation of the Lorentz group using harmonic oscillator wave functions. According to his 1927 paper, the Gaussian cut-off along the time axis represents the time-energy uncertainty relation. His c-number relation means that only the ground state is allowed along the time axis (Kim and Noz 1986).

On the other hand, this expression is consistent with his earlier papers on the time-energy uncertainty relation (Dirac 1927). In those papers, Dirac observed that there is a time-energy uncertainty relation, while there are no excitations along the time axis.

Let us look at Fig. 5.2 carefully. This figure is a pictorial representation of Dirac’s Eq. (5.2), with localization in both space and time coordinates. Then Dirac’s fundamental question would be how this figure appears to a moving observer.

## 5.2 Dirac’s forms of relativistic dynamics

In 1949, the Reviews of Modern Physics published a special issue to celebrate Einstein’s 70<sup>th</sup> birthday. This issue contains Dirac’s paper entitled “Forms of Relativistic Dynamics” (Dirac 1949). In this paper, Dirac points out problems that have to be overcome in extending quantum mechanics to be relativistic. He considers possible constraints that can be imposed on the Lorentz transformations.

1. In this paper also, Dirac’s main problem was to deal with the time variable, and he writes down the equation

$$x_0 \approx 0. \quad (5.3)$$

Since Dirac offered no further explanations on his  $\approx$  sign, we are free to interpret this as his c-number time-energy uncertainty relation. In terms of the harmonic oscillators which he discussed in 1945, this condition means that the harmonic oscillator wave function in the time variable is always in the ground state.

2. Dirac wrote down ten generators of the Poincaré group and their commutation relations, which he called Poisson brackets. The Poincaré group is the inhomogeneous

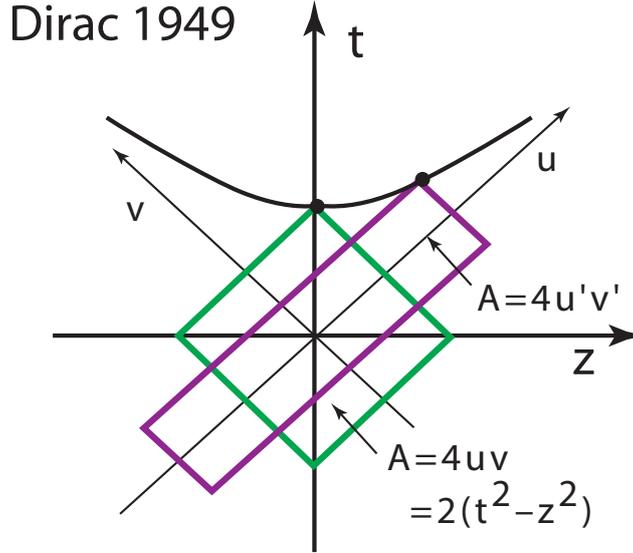


Figure 5.3: Lorentz boost in the light-cone coordinate system. The boost traces a point along the hyperbola. The boost also squeezes the square into a rectangle (Kim and Noz 1986).

Lorentz group with three rotation generators, three boost generators, and four generators of translations. Dirac notes that the rotation and translation generators are associated with angular momentum and space-time translations respectively. On the other hand, there are no dynamical variables associated with the generators of boosts.

Dirac then states that the construction of relativistic quantum mechanics is achieved through the construction of the representation of the Poincaré group. Even though Dirac did not tell us how to construct such a representation, he made an attempt in 1945 (Dirac 1945) to use harmonic oscillators to construct a representation of the Lorentz group. We should therefore use harmonic oscillator wave functions to construct the representation of the Poincaré group and the Lorentz-covariant quantum mechanics.

3. Also in his 1949 paper, Dirac introduced the light-cone coordinate system. When the system is boosted along the  $z$  direction, the transformation takes the form

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}. \quad (5.4)$$

This is not a rotation, because the space and time variables become entangled. Dirac introduced his light-cone variables defined as

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}, \quad (5.5)$$

hence the boost transformation of Eq. (5.4) takes the form

$$u' = e^\eta u, \quad v' = e^{-\eta} v. \quad (5.6)$$

The  $u$  variable becomes expanded while the  $v$  variable becomes contracted, as is illustrated in Fig. 5.3. Their product

$$uv = \frac{1}{2}(z+t)(z-t) = \frac{1}{2}(z^2 - t^2) \quad (5.7)$$

remains invariant. Indeed, in Dirac's picture, the Lorentz boost is a an area-preserving squeeze transformation, as indicated in Fig. 5.3.

Let us go back to the ground-state wave function of Eq.(5.2) whose Gaussian form is illustrated in Fig. 5.2. If the oscillator moves along the  $z$  direction, it should appear as

$$\exp \left\{ -\frac{1}{4} \left[ e^{-2\eta}(z+t)^2 + e^{2\eta}(z-t)^2 \right] \right\}, \quad (5.8)$$

according to the Lorentz-boost in the light-cone system given in Eq.(5.6). This is a Lorentz-squeezed Gaussian distribution, as shown in Fig. 5.4. Indeed. the Lorentz boost is a squeeze transformation (Kim and Noz 1973).

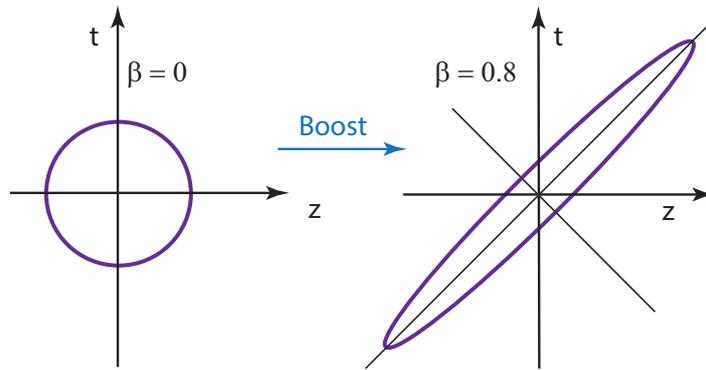


Figure 5.4: Lorentz-squeezed quantum mechanics. Dirac's attempt for relativistic quantum mechanics starts from the Gaussian distribution given in Fig. 5.2. This figure shows how the Gaussian distribution appears to an observer moving with the velocity parameter  $\beta = v/c = 0.8$  (Kim *et al.* 1979a)

### 5.3 Running waves and standing waves

We are quite familiar with the Klein-Gorden equation

$$\left[ \left( \frac{\partial}{\partial x} \right)^2 + m^2 \right] \phi(x) = 0, \quad (5.9)$$

for a single particle, where

$$\left( \frac{\partial}{\partial x} \right)^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \quad (5.10)$$

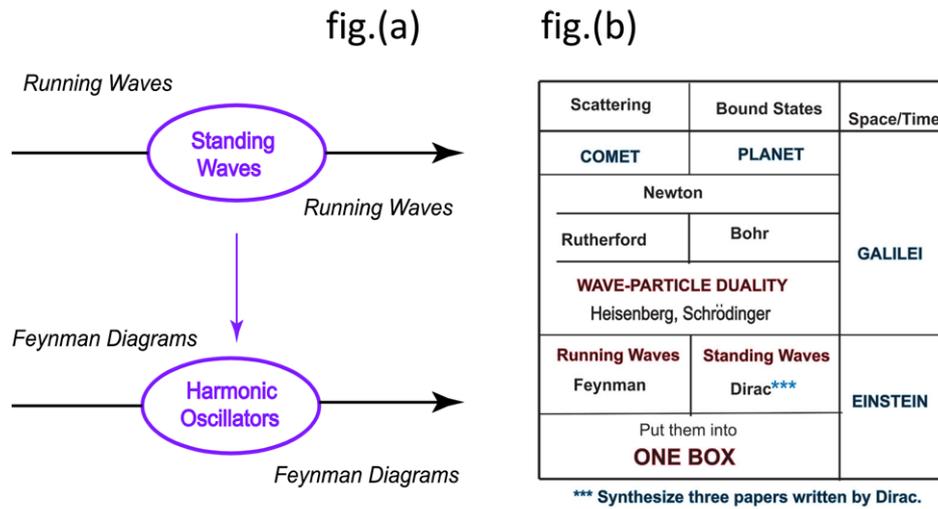


Figure 5.5: Feynman’s suggestion for combining quantum mechanics with special relativity. Feynman diagrams work for running waves, and they provide a satisfactory interpretation of scattering states in Einstein’s world. For standing waves trapped inside a hadron, Feynman (Feynman 1970) suggested harmonic oscillators as the first step, as illustrated in fig.(a). From his suggestion, we can construct a historical perspective starting from comets and planets, as shown in fig.(b). Dirac’s three papers are discussed in Sec. 5.2. It has been shown by Han *et al.* that the covariant oscillator formalism shares the same set of physical principles as Feynman diagrams for scattering problems (Han *et al.* 1981).

This equation and its solutions are well know in physics. Then the equation

$$\left[ \left( \frac{\partial}{\partial x_a} \right)^2 + \left( \frac{\partial}{\partial x_b} \right)^2 + m_a^2 + m_b^2 \right] \phi(x_a) \phi(x_b) = 0, \tag{5.11}$$

is for two independent particles whose coordinates are  $x_a$  and  $x_b$  respectively. They are four-vectors, and we shall use the notation

$$x^2 \equiv t^2 - z^2 - x^2 - y^2 \tag{5.12}$$

for both  $x_a$  and  $x_b$ . The physics of this system for two free particles is also well known. The question is what happens when  $(m_a^2 + m_b^2)$  is replaced by a term containing  $(x_a - x_b)^2$ .

Indeed, Feynman *et al.* (Feynman *et al.* 1971) wrote down the equation

$$\left\{ 2 \left[ \left( \frac{\partial}{\partial x_a} \right)^2 + \left( \frac{\partial}{\partial x_b} \right)^2 \right] - \frac{1}{16} (x_a - x_b)^2 + m_0^2 \right\} \phi(x_a, x_b) = 0, \tag{5.13}$$

for the bound state called the “hadron” consisting of two constituent particles called “quarks” bound together in a harmonic oscillator potential.

Let us introduce the coordinates

$$X = \frac{x_a + x_b}{2}, \quad \text{and} \quad x = \frac{x_a - x_b}{2\sqrt{2}}. \tag{5.14}$$

The  $X$  coordinate is for the space-time specification of the hadron, while the  $x$  variable measures the relative space-time separation between the quarks. In terms of these variables, Eq. (5.13) can be written as

$$\left\{ \left( \frac{\partial}{\partial X} \right)^2 + m_0^2 + \frac{1}{2} \left[ \left( \frac{\partial}{\partial x} \right)^2 - x^2 \right] \right\} \phi(X, x) = 0, \quad (5.15)$$

This equation is separable in the  $X$  and  $x$  variables, and can be separated by using the equation:

$$\phi(X, x) = f(X) \psi(x), \quad (5.16)$$

where  $f(X)$  and  $\psi(x)$  satisfy the following differential equations respectively:

$$\left\{ \left( \frac{\partial}{\partial X} \right)^2 + m_0^2 + (\lambda + 1) \right\} f(X) = 0, \quad (5.17)$$

and

$$\frac{1}{2} \left[ \left( \frac{\partial}{\partial x} \right)^2 - x^2 \right] \psi(x) = (\lambda + 1) \psi(x). \quad (5.18)$$

The differential equation of Eq. (5.17) is a Klein-Gordon equation, and its solutions are well known. It takes the form

$$f(X) = \exp(\pm ip \cdot X) \quad (5.19)$$

with

$$p^2 = m_0^2 + (\lambda + 1), \quad (5.20)$$

where  $p$  is the four-momentum of the hadron.  $p^2$  is, of course, the  $(mass)^2$  of the hadron and is numerically constrained to take the values allowed by Eq. (5.20). The separation constant  $\lambda$  is determined from the solutions of the harmonic oscillator equation of Eq. (5.18).

Indeed, the wave function of Eq. (5.16) is for the hadron moving with the four-momentum  $p_\mu$  with the internal structure dictated by the oscillator equation, as is described in Fig. 5.5. Wigner's little group is applicable to the internal space-time symmetry dictated by the oscillator equation of Eq. (5.18) (Kim and Noz 1978, Kim *et al.* 1979a, Kim and Noz 1986).

The space-time transformation of the total wave function of Eq. (5.16) is generated by the following ten generators of the Poincaré group. The operators

$$P_\mu = i \frac{\partial}{\partial X^\mu} \quad (5.21)$$

generate space-time translations. Lorentz transformations, which include boosts and rotations, are generated by

$$M_{\mu\nu} = L_{\mu\nu}^* + L_{\mu\nu} \quad (5.22)$$

where

$$L_{\mu\nu}^* = i \left( X_\mu \frac{\partial}{\partial X^\nu} - X_\nu \frac{\partial}{\partial X^\mu} \right),$$

$$L_{\mu\nu} = i \left( x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \right).$$

The translation operators  $P_\mu$ , act only on the hadronic coordinate, and do not affect the internal coordinate. The operators  $L_{\mu\nu}^*$  and  $L_{\mu\nu}$  Lorentz-transform the hadronic and internal coordinates respectively. The above ten generators satisfy the commutation relations for the Poincaré group.

In order to consider irreducible representations of the Poincaré group, we have to construct wave functions which are diagonal in the invariant Casimir operators of the group, which commute with all the generators of Eqs. (5.21) and (5.22). The Casimir operators in this case are

$$P^\mu P_\mu, \quad \text{and} \quad W^\mu W_\mu, \quad (5.23)$$

where

$$W_\mu = \epsilon_{\mu\nu\alpha\beta} P^\nu M^{\alpha\beta} \quad (5.24)$$

The eigenvalues of the above  $P^2$  and  $W^2$  represent respectively the mass and spin of the hadron.

The algebra of these generators becomes much simpler when the hadronic momentum is constant, as in the case of Wigner's little group. While translation generators can be dropped from the algebra, the operator  $P^\nu$  can be replaced by the four-vector

$$p = (E, p, 0, 0) \quad (5.25)$$

for the hadron momentum moving in the  $z$  direction. As a consequence the eigenvalues of the Casimir operators become  $m^2 = (mass)^2$  and  $m^2\ell(\ell + 1)$ , where  $\ell$  is the total angular momentum of oscillator. These eigenvalues are invariant under Poincaré or Lorentz transformations.

## 5.4 Little groups for relativistic extended particles

The harmonic oscillator equation of Eq. (5.18) is invariant under Lorentz transformations. For instance, if the system is boosted along the  $z$  direction according to Eq. (5.4), the differential equation takes the same form in the new coordinate variables. Thus, the solution also takes the previous form. With this point in mind, we can now study the solution of the differential equation in the Lorentz frame where the hadron is at rest. Let us spell out the oscillator equation.

$$\frac{1}{2} \left\{ \left[ x^2 + y^2 + z^2 - \left( \frac{\partial}{\partial x} \right)^2 - \left( \frac{\partial}{\partial y} \right)^2 - \left( \frac{\partial}{\partial z} \right)^2 \right] - \left[ t^2 - \left( \frac{\partial}{\partial t} \right)^2 \right] \right\} \psi(x) = (\lambda + 1)\psi(x). \quad (5.26)$$

According to Dirac's c-number time-energy uncertainty relation, the time component of the solution should be always in the ground state, and thus the solution takes the form

$$\psi(x) = \varphi(z, x, y) \left[ \left( \frac{1}{\pi} \right)^{1/4} e^{-t^2/2} \right], \quad (5.27)$$

with

$$\frac{1}{2} \left[ x^2 + y^2 + z^2 - \left( \frac{\partial}{\partial x} \right)^2 - \left( \frac{\partial}{\partial y} \right)^2 - \left( \frac{\partial}{\partial z} \right)^2 \right] \varphi(z, x, y) = \left( \lambda + \frac{3}{2} \right) \varphi(z, x, y). \quad (5.28)$$

This equation is very familiar to us from textbooks. However, the equation carried the following additional interpretations.

1. The Cartesian variables  $z$ ,  $x$ , and  $y$  are internal coordinate variables for the hadron.
2. This equation is separable in both the spherical and Cartesian coordinate system. For the discussion of the Poincaré symmetry, we need the spherical coordinate system to construct the representation diagonal in the Casimir operators where the eigenvalue  $\ell$  is needed.
3. When the system is boosted along the  $z$  direction, the transverse  $x$  and  $y$  are not affected, and they can be separated out from the differential equation of Eq. (5.26).
4. The spherical solutions can be written in terms of the linear combinations of the Cartesian solutions.

The solution in the spherical coordinate system should take the form

$$\varphi(r, \theta, \phi) = R_{n\ell m}(r)Y_{\ell m}(\theta, \phi), \quad (5.29)$$

where  $Y_{\ell m}(\theta, \phi)$  are the spherical harmonics. The radial function  $R_{n\ell m}(r)$  is well defined, but its explicit form is not readily available in the literature. It should take the form

$$R_{n\ell m}(r) = r^m g_{n\ell}(r)e^{-r^2/2}, \quad (5.30)$$

with

$$g_{n\ell} = \sum_k a_{2k} r^{2k}, \quad (5.31)$$

where

$$\frac{a_{2k+2}}{a_{2k}} = \frac{2(\lambda - \ell - 2k)}{\ell(\ell + 1) - (\ell + 2k + 3)(\ell + 2k + 2)}. \quad (5.32)$$

For large values of  $k$ , this ratio becomes  $1/k$ , which is like the expansion of the exponential  $\exp(r^2)$  leading the radial function of Eq. (5.30) to increase as  $\exp(r^2/2)$ . Thus the series has to be truncated with

$$\lambda = 2k + \ell. \quad (5.33)$$

The first term  $a_0$  in the series is determined by the normalization condition

$$\int_0^\infty [rR_{n\ell m}(r)]^2 dr = 1. \quad (5.34)$$

The increases in  $\ell$  and  $n$  are called the orbital and radial excitations in the literature.

If the system is Lorentz-boosted along the  $z$  direction according to Eq. (5.4), the Lorentz-invariant differential equation of Eq. (5.18) remains invariant. The  $z$  and  $t$  variables in Eq. (5.26) become  $z'$  and  $t'$  respectively, and the wave function becomes modified accordingly. The important point is that the eigenvalue  $\lambda$  remains invariant.

## 5.5 Further properties of covariant oscillator wave functions

Since the  $x$  and  $y$  coordinates are not affected, we drop their terms from the differential equation from Eq. (5.18), and consider the equation

$$\frac{1}{2} \left\{ \left[ z^2 - \left( \frac{\partial}{\partial z} \right)^2 \right] - \left[ t^2 - \left( \frac{\partial}{\partial t} \right)^2 \right] \right\} \psi(z, t) = \lambda \psi(z, t). \quad (5.35)$$

The solution of this equation should take the form

$$\psi_0^n(z, t) = \left[ \frac{1}{\pi 2^n n!} \right]^{1/2} H_n(z) \exp \left[ -\frac{1}{2} (z^2 + t^2) \right], \quad (5.36)$$

where  $H_n(z)$  is the Hermite polynomial of order  $n$ .

The differential equation of Eq. (5.35) is invariant under the Lorentz boost along the  $z$  direction, and is invariant under the replacements of  $z$  and  $t$  by  $z'$  and  $t'$  respectively, where

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \eta & -\sinh \eta \\ -\sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}. \quad (5.37)$$

This is the inverse of the transformation given in Eq. (5.4). We thus achieve the Lorentz boost of the wave function by writing

$$\begin{aligned} \psi_\eta^n(z, t) &= \psi_0^n(z', t') \\ &= \left[ \frac{1}{\pi 2^n n!} \right]^{1/2} H_n(z') \exp \left[ -\frac{1}{2} (z'^2 + t'^2) \right]. \end{aligned} \quad (5.38)$$

It is possible to expand this in terms of the functions of  $z$  and  $t$  variables (Ruiz 1974, Kim *et al.* 1979b, Kim and Noz 1986, Rotbart 1981), and the result is

$$\psi_\eta^n(z, t) = \left( \frac{1}{\cosh \eta} \right)^{(n+1)} \sum_k \left[ \frac{(n+k)!}{n!k!} \right]^{1/2} (\tanh \eta)^k \chi_{n+k}(z) \chi_k(t), \quad (5.39)$$

where  $\chi_n(z)$  is the  $n^{\text{th}}$  excited state oscillator wave function which takes the familiar form

$$\chi_n(z) = \left[ \frac{1}{\sqrt{\pi} 2^n n!} \right]^{1/2} H_n(z) \exp \left( -\frac{z^2}{2} \right). \quad (5.40)$$

If  $n = 0$ , this formula becomes simplified to (Ruiz 1974)

$$\psi_\eta^0(z, t) = \left( \frac{1}{\cosh \eta} \right)^{1/2} \sum_k (\tanh \eta)^k \chi_k(z) \chi_k(t). \quad (5.41)$$

This formula plays an important role also in squeezed states of light (Yuen 1976) and also in continuous-variable entanglement (Giedke *et al.* 2003, Dodd and Halliwell 2004, Braunstein and van Loock 2005, Adesso and Illuminati 2007, Paz and Roncaglia 2008, Chou *et al.* 2008, Xiang *et al.* 2011).

According to the Gaussian form of Eq.(5.8), the ground-state wave function of Eq. (5.41) takes the form (Kim and Noz 1973)

$$\psi_{\eta}^0(z, t) = \left(\frac{1}{\pi}\right)^{1/2} \exp \left\{ -\frac{1}{4} \left[ e^{-2\eta}(z+t)^2 + e^{2\eta}(z-t)^2 \right] \right\}. \quad (5.42)$$

This is precisely the formula for Dirac's picture of the Lorentz squeeze, as illustrated in Fig. 5.2.

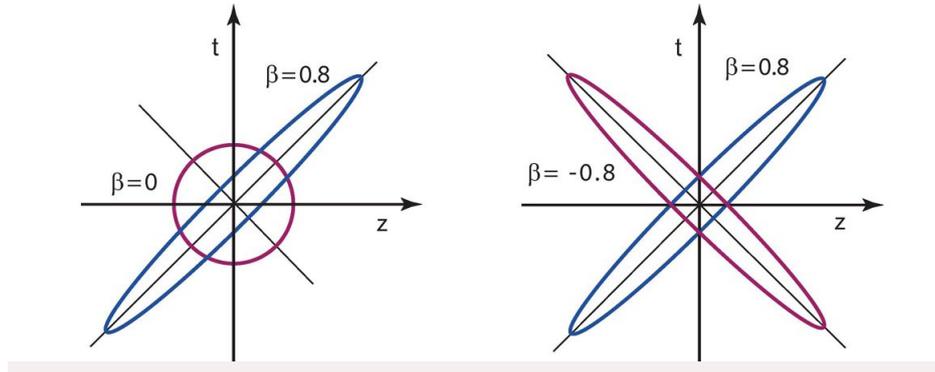


Figure 5.6: Two overlapping wave functions. The wave functions with  $\beta = 0$  and  $\beta = 0.08$  are sketched in Fig. 5.4. They can overlap as shown in the present figure. The wave functions can also move in the opposite directions with  $\beta = \pm 0.8$  (Kim and Noz 1973, 1986).

Relativity and quantum mechanics are the two most important physical theories developed in the 20th century. Thus, the most pressing issue in modern physics is to make quantum mechanics consistent with special relativity and then with general relativity. Since the harmonic oscillator plays the central role in quantum mechanics, the problem is to construct harmonic oscillator wave functions consistent with Lorentz covariance. A comprehensive list of early papers on this problem was given by Kim and Noz in their book (Kim and Noz 1986). There also more recent papers on this subject (Navarro and Navarro-Salas 1996, Gazeau and Graffi 1997, Oda *et al.* 1999, Bars 2009, Kowalski and Rembielinski 2010, Simon 2011, Pavsc 2013).

## 5.6 Lorentz contraction of harmonic oscillators

Let us now consider the overlap of the wave function  $\psi_{\eta}^n(z, t)$  with that with  $\eta = 0$ , as indicated in Fig. 5.6. We are interested in the integral

$$\int \left( \psi_{\eta}^n(z, t) \right)^* \psi_0^n(z, t) dz dt, \quad (5.43)$$

where  $\psi_0^n(z, t)$ , given in Eq. (5.36), can be written as

$$\psi_0^n = \chi_n \chi_0(t). \quad (5.44)$$

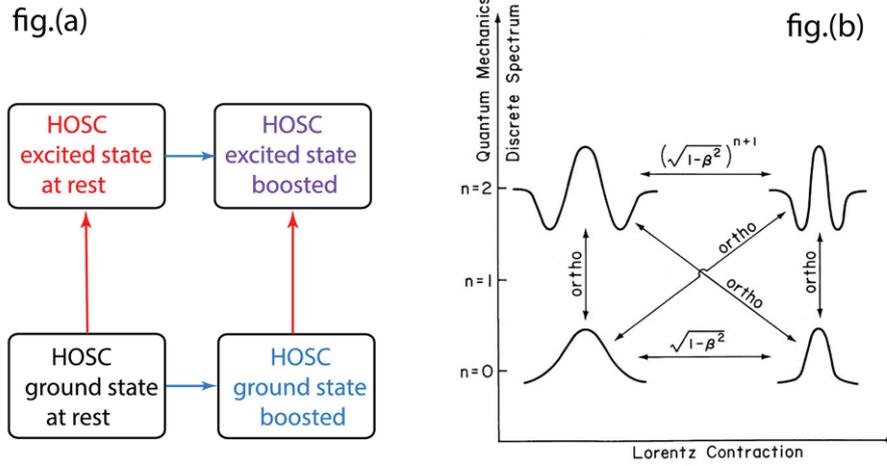


Figure 5.7: Orthogonality relations for covariant oscillator wave functions. The harmonic oscillator can be excited and can also be Lorentz boosted as illustrated in fig.(a). The orthogonality relations remain invariant under Lorentz boosts, and their inner products have the Lorentz-contraction property given in fig.(b) (Ruiz 1974, Kim and Noz 1986).

Then evaluation of this integral is straight-forward from Eq. (5.39), and the result is (Ruiz 1974)

$$\left(\psi_{\eta}^{n'}, \psi_0^n\right) = \left(\frac{1}{\cosh \eta}\right)^{(n+1)} \delta_{nn'} \quad (5.45)$$

This means that the orthogonality relation is preserved between two wave functions in two different frames.

If  $n = n'$ , the inner product between two wave functions leads to the contraction given on the right hand side of Eq. (5.45). In terms of the velocity parameter  $\beta = v/c$ , where  $v$  is the hadronic velocity,

$$\frac{1}{\cosh \eta} = \sqrt{1 - \beta^2}. \quad (5.46)$$

This expression is more familiar to us, and the right-hand side of Eq. (5.45) is

$$\left(\sqrt{1 - \beta^2}\right)^{(n+1)}. \quad (5.47)$$

For the ground-state wave function with  $n = 0$ , it is like the Lorentz contraction of a rigid body. For the first excited state, it is like an additional rod. This is not surprising in view of the fact that the excited states are obtained through application of the step-up operator. The  $n^{\text{th}}$  excited state  $|n\rangle$  can be written as

$$\frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle, \quad (5.48)$$

The additional contraction factor  $\sqrt{1 - \beta^2}$  comes from the step-up operator.

If the value of  $\eta$  in one of the wave functions is replaced by nonzero value  $\eta'$ ,  $\cosh \eta$  in Eq. (5.39) should become  $\cosh(\eta - \eta')$ . Of particular interest is the case with  $\eta' = -\eta$ ,

as shown in Fig.(5.6). In this case, this is an overlap of two wave functions moving in the opposite directions, and the contaction factor is

$$\left(\frac{1}{\cosh(2\eta)}\right)^{(n+1)}, \quad (5.49)$$

which, in the language of  $\beta$ , becomes

$$\left(\frac{1-\beta^2}{1+\beta^2}\right)^{(n+1)}. \quad (5.50)$$

Based on these orthogonality and contraction properties, it appears possible to give a quantum probability interpretation to the covariant harmonic oscillator in the Lorentz-covariant world as shown in Fig.(5.7). On the other hand, these wave functions contain the time separation variable between the constituent particles. What is the meaning of the distribution in terms of this variable, which is thoroughly hidden (Kim and Noz 2003) in the present form of quantum mechanics? In order to clarify this issue, let us examine the concept of Feynman's rest of the universe.

## 5.7 Feynman's rest of the universe

As was noted in the previous section, the time-separation variable has an important role in the covariant formulation of the harmonic oscillator wave functions. It should exist wherever the space separation exists. The Bohr radius is the measure of the separation between the proton and electron in the hydrogen atom. If this atom moves, the radius picks up the time separation, according to Einstein.

On the other hand, the present form of quantum mechanics does not include this time-separation variable. The best way we can interpret it at the present time is to treat this time separation as a variable in Feynman's rest of the universe (Han *et al.* 1999). In his book on statistical mechanics (Feynman 1972), Feynman states

*When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

The failure to include what happens outside the system results in an increase of entropy. The entropy is a measure of our ignorance and is computed from the density matrix (von Neumann 1932). The density matrix is needed when the experimental procedure does not analyze all relevant variables to the maximum extent consistent with quantum mechanics (Fano 1957). If we do not take into account the time-separation variable, the result is an increase in entropy (Kim and Wigner 1990, Kim and Noz 2014).

For the covariant oscillator wave functions defined in Eq. (5.38), the pure-state density matrix is

$$\rho_\eta^n(z, t; z', t') = \psi_\eta^n(z, t)\psi_\eta^n(z', t'), \quad (5.51)$$

which satisfies the condition  $\rho^2 = \rho$  :

$$\rho_\eta^n(z, t; x', t') = \int \rho_\eta^n(z, t; x'', t'') \rho_\eta^n(z'', t''; z', t') dz'' dt'' . \quad (5.52)$$

However, in the present form of quantum mechanics, it is not possible to take into account the time separation variables. Thus, we have to take the trace of the matrix with respect to the  $t$  variable. Then the resulting density matrix is

$$\begin{aligned} \rho_\eta^n(z, z') &= \int \psi_\eta^n(z, t) \psi_\eta^n(z', t) dt \\ &= \left( \frac{1}{\cosh \eta} \right)^{2(n+1)} \sum_k \frac{(n+k)!}{n!k!} (\tanh \eta)^{2k} \psi_{n+k}(z) \psi_{n+k}^*(z'). \end{aligned} \quad (5.53)$$

The trace of this density matrix is one, but the trace of  $\rho^2$  is less than one, as

$$\begin{aligned} Tr(\rho^2) &= \int \rho_\eta^n(z, z') \rho_\eta^n(z', z) dz dz' \\ &= \left( \frac{1}{\cosh \eta} \right)^{4(n+1)} \sum_k \left[ \frac{(n+k)!}{n!k!} \right]^2 (\tanh \eta)^{4k}, \end{aligned} \quad (5.54)$$

which is less than one. This is due to the fact that we do not know how to deal with the time-like separation in the present formulation of quantum mechanics. Our knowledge is less than complete.

The standard way to measure this ignorance is to calculate the entropy defined as

$$S = -Tr(\rho \ln(\rho)). \quad (5.55)$$

If we pretend to know the distribution along the time-like direction and use the pure-state density matrix given in Eq. (5.51), then the entropy is zero. However, if we do not know how to deal with the distribution along  $t$ , then we should use the density matrix of Eq. (5.53) to calculate the entropy, and the result is

$$\begin{aligned} S &= (n+1) \left\{ (\cosh \eta)^2 \ln(\cosh \eta)^2 - (\sinh \eta)^2 \ln(\sinh \eta)^2 \right\} \\ &\quad - \left( \frac{1}{\cosh \eta} \right)^{2(n+1)} \sum_k \frac{(n+k)!}{n!k!} \ln \left[ \frac{(n+k)!}{n!k!} \right] (\tanh \eta)^{2k}. \end{aligned} \quad (5.56)$$

In terms of the velocity paramter  $\beta$  of the hadron,

$$\tanh \eta = \beta = \frac{v}{c}, \quad (5.57)$$

and

$$\begin{aligned} S &= -(n+1) \left\{ \ln(1-\beta^2) + \frac{\beta^2 \ln \beta^2}{1-\beta^2} \right\} \\ &\quad - (1-\beta^2)^{(n+1)} \sum_k \frac{(n+k)!}{n!k!} \ln \left[ \frac{(n+k)!}{n!k!} \right] \beta^{2k}. \end{aligned} \quad (5.58)$$

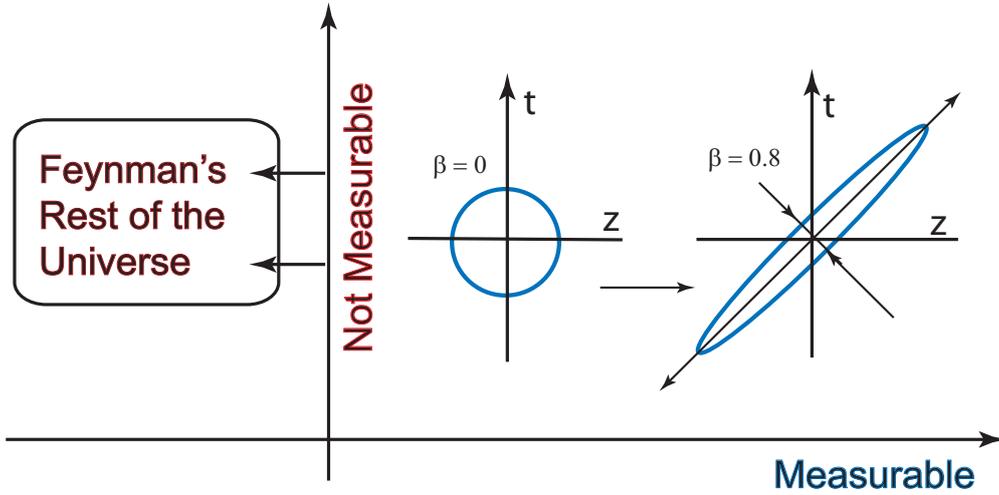


Figure 5.8: Localization property in the  $zt$  plane. When the hadron is at rest, the Gaussian form is concentrated within a circular region specified by  $(z+t)^2 + (z-t)^2 = 1$ . As the hadron gains speed, the region becomes deformed to  $e^{-2\eta}(z+t)^2 + e^{2\eta}(z-t)^2 = 1$ . Since it is not possible to make measurements along the  $t$  direction, we have to deal with information that is less than complete (Kim and Wigner 1990).

Let us go back to the wave function given in Eq. (5.39). As is illustrated in Fig. 5.8, its localization property is dictated by the Gaussian factor which corresponds to the ground-state wave function. For this reason, we expect that much of the behavior of the density matrix or the entropy for the  $n^{\text{th}}$  excited state will be the same as that for the ground state with  $n = 0$ .

For the ground state with  $n = 0$ , the density matrix can be computed from the Gaussian form of Eq. (5.41), and it becomes

$$\rho(z, z') = \left( \frac{1}{\pi \cosh(2\eta)} \right)^{1/2} \exp \left\{ -\frac{1}{4} \left[ \frac{(z+z')^2}{\cosh(2\eta)} + (z-z')^2 \cosh(2\eta) \right] \right\}, \quad (5.59)$$

For this ground state, the entropy becomes

$$S = (\cosh \eta)^2 \ln(\cosh \eta)^2 - (\sinh \eta)^2 \ln(\sinh \eta)^2. \quad (5.60)$$

In terms of the velocity parameter  $\beta$ , this entropy can be written as

$$S = \frac{1}{1-\beta^2} \ln \left[ \frac{1}{1-\beta^2} \right] - \beta^2 \ln \beta^2. \quad (5.61)$$

The width of the distribution becomes  $\sqrt{\cosh(2\eta)}$ . In terms of  $\beta$ ,

$$\sqrt{\cosh(2\eta)} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}, \quad (5.62)$$

which becomes very large as  $\beta \rightarrow 1$ , as is illustrated in Fig. 5.9

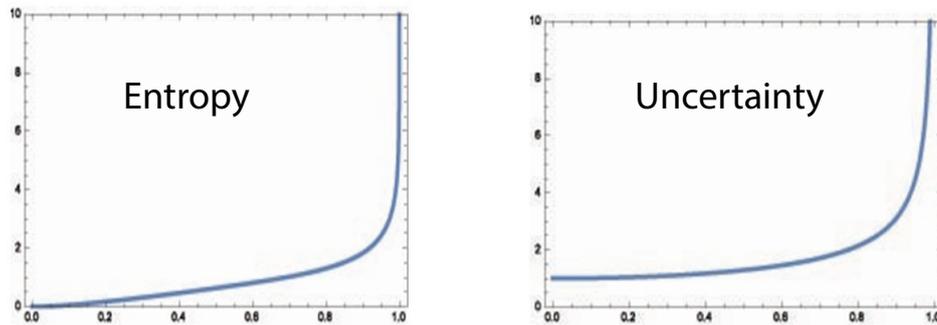


Figure 5.9: Entropy and uncertainty in the Lorentz-covariant system. The horizontal axis measures  $\beta$  in both graphs. The formulas for the entropy and uncertainty are given in Eq. (5.60) and Eq. (5.65) respectively. These quantities arise from our ignorance about the time-separation variable, which is hidden in the present form of quantum mechanics (Kim and Noz 2014).

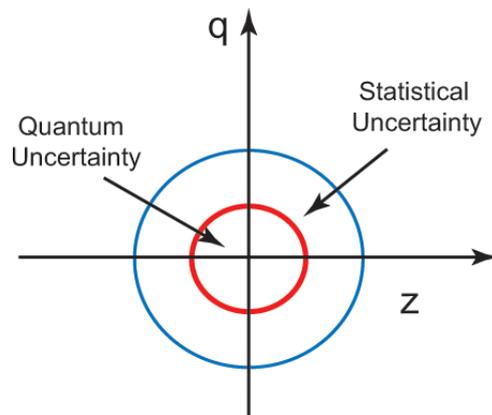


Figure 5.10: The uncertainty from the hidden time-separation coordinate. The small circle indicates the minimal uncertainty when the hadron is at rest. More uncertainty is added when the hadron moves. This is illustrated by a larger circle. The radius of this circle increases by  $\sqrt{\cosh(2\eta)}$  (Kim and Noz 2014) .

The time-separation variable exists in the Lorentz-covariant world, but we pretend not to know about it. It thus is in Feynman's rest of the universe. If we do not measure this time separation, it becomes translated into the entropy.

We can see the uncertainty in our measurement process from the Wigner function defined as (Wigner 1932, Kim and Noz 1991)

$$W(z, q) = \frac{1}{\pi} \int \rho(z + y, z - y) e^{2iqy} dy. \quad (5.63)$$

After integration, this Wigner function becomes

$$W(z, q) = \frac{1}{\pi \cosh(2\eta)} \exp \left\{ - \left( \frac{z^2 + q^2}{\cosh(2\eta)} \right) \right\}. \quad (5.64)$$

We use here  $q$  instead of  $p$  for the momentum conjugate to  $z$ . The notation  $p$  has been used for the hadronic momentum.

This Wigner phase-space distribution is illustrated in Fig. 5.10. The smaller inner circle corresponds to the minimal uncertainty of the single oscillator. The larger circle is for the total uncertainty including the statistical uncertainty from our failure to observe the time-separation variable. The larger radius is

$$\sqrt{\cosh(2\eta)} = \sqrt{\frac{1 + \beta^2}{1 - \beta^2}}. \quad (5.65)$$

The behavior of this radius is illustrated in Fig. 5.9. This radius takes the minimum value of one when  $\beta = 0$ , and increases rapidly when  $\beta$  becomes close to one.

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# Chapter 6

## Quarks and partons in the Lorentz-covariant world

Since the time of Einstein and Bohr, there has been an evolution of the way in which we look at quantum bound states, as illustrated in Fig. 6.1. The evolution took place in the following three steps.

1. The energy levels of the hydrogen atom played the pivotal role by replacing the electron orbit of the hydrogen atom with a standing wave, leading to bound states in quantum mechanics. However, the hydrogen atom cannot play a role in the Lorentz covariant world since it cannot be accelerated to a relativistic speed.
2. 1964, the proton became a bound state of the more fundamental constituents called “quarks” (Gell-mann 1964). Of course, the proton is different from the hydrogen atom, but inherits the same quantum mechanics from the hydrogen atom. Unlike the hydrogen atom, the proton can be accelerated, and its speed can become very close to that of light. Thus, it is possible to study the quantum mechanics of the hydrogen atom or bound states in the Lorentz-covariant world by studying the proton in the quark model. Fig. 6.1 illustrates this transition.
3. In 1969, Feynman observed that the proton, when it moves with a velocity close to that of light, appears like a collection of partons with a wide-spread momentum distribution (Feynman 1969). Partons are like free particles. Quarks and partons are the same particles but they appear differently to observers in two different reference frames. Therefore, there must be a Lorentz-covariant model for quantum bound states, as illustrated in Fig. 6.2.

At the time of Einstein and Bohr, both the proton and electron were regarded as point particles. However, the discovery of Hofstadter and McAllister changed our picture of the proton (Hofstadter and McAllister 1955). The proton charge has an internal distribution. Within the framework of quantum electrodynamics, it is possible to calculate the Rutherford formula for electron-proton scattering when both electron and proton are point particles. Because the proton is not a point particle, there is a deviation from the Rutherford formula. We describe this deviation as the “proton form factor” which depends on the momentum transfer during electron-proton scattering.

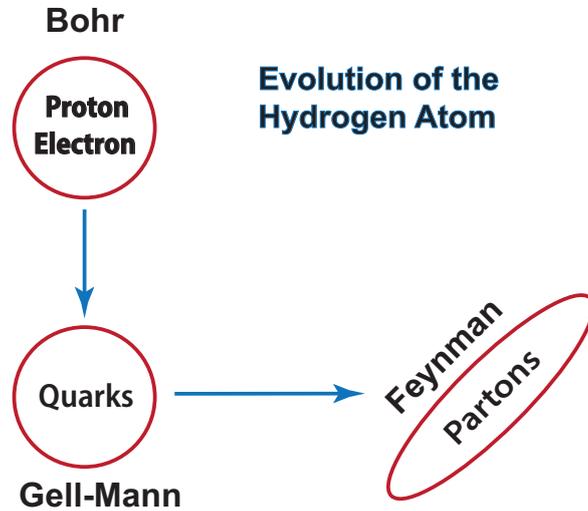


Figure 6.1: Evolution of the hydrogen atom. The electron orbit was replaced by a standing wave, but the hydrogen atom cannot not be accelerated. In 1964, the proton as a bound state inherited the quantum mechanics of the hydrogen atom (Gell-mann 1964). The proton these days can move with a speed very close to that of light and exhibits the properties of quantum bound states in the Lorentz-covariant world.

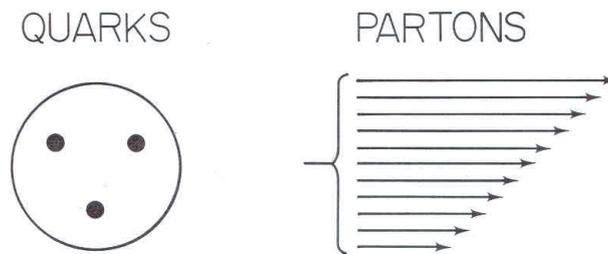


Figure 6.2: Two distinct ways in which the proton appears in the real world. If the proton is at rest, it appears as a bound state of three quarks (Gell-mann 1964). If it moves with a speed close to that of light, it appears like a collection of an infinite number of partons (Feynman 1969). Then the question is whether quarks and partons are two different manifestations of the same Lorentz-covariant entity.

Indeed, the study of the proton form factor has been and still is one of the central issues in high-energy physics. The proton form factor decreases as the momentum transfer increases. Its behavior is called the “dipole cut-off” meaning an inverse-square decrease, and it has been a challenging problem in quantum field theory and other theoretical models (Frazer and Fulco 1960). Since the emergence of the quark model in 1964 (Gell-Mann 1964), the hadrons are regarded as quantum bound states of quarks with space-time extensions.

Furthermore, the hadronic mass spectra indicate that the binding force between the quarks is like that of the harmonic oscillator (Feynman *et al.* 1971). This leads us to suspect that the quark model with harmonic oscillator wave functions could explain the behavior of the proton form factor. There are indeed many papers written on this subject. We shall return to this problem in Sec. 6.4.

Another problem in high-energy physics is Feynman’s parton picture (Feynman 1969). If the hadron is at rest, we can approach this problem within the framework of bound-state quantum mechanics. If it moves with a speed close to that of light, it appears as a collection of an infinite number of partons, which interact with external signals incoherently. This phenomenon raises the question of whether the Lorentz boost destroys quantum coherence (Kim 1998).

## 6.1 Lorentz-covariant quark model

In the quark model, mesons are two-body bound states of one quark and one anti-quark, and baryons are bound states of three quarks. The early successes of the quark model include the ratio of the proton-neutron electromagnetic potential and magnetic moments (Beg *et al.* 1964). Also the hadronic mass spectra are like those of three-dimensional harmonic oscillators (Feynman *et al.* 1971).

The question then is how the mass spectrum calculated within the framework of non-relativistic quantum mechanics is valid for this relativistic case, while ignoring the time-separation variable. For this question, the answer given in the 1971 paper of Feynman *et al.* is not satisfactory. The correct answer to this question is that Wigner’s little group for massive particles is like the three-dimensional rotation group as was spelled out in Chapter 2 and in Chapter 5. The role of the time-separation variable is discussed there.

Our original question is how the hydrogen atom looks to a moving observer. The question now is how much we can learn about this Bohr-Einstein issue by studying the proton in the quark model based on the three-dimensional harmonic oscillator. For the hydrogen atom, we use the Coulomb potential, while the binding force between quarks is that of the oscillator. The point is that those two different potentials share the same quantum mechanics.

For this purpose, we will need a bound-state wave function which can be Lorentz-boosted. Here the natural choice is the harmonic oscillator wave function discussed in Chapter 5. We can start with the ground-state wave function which can be Lorentz boosted. Since the harmonic oscillator wave function is separable in the Cartesian coordinate system, we can leave out the transverse components of the wave function, and consider only the longitudinal and time-like coordinates. For this purpose, let us rewrite

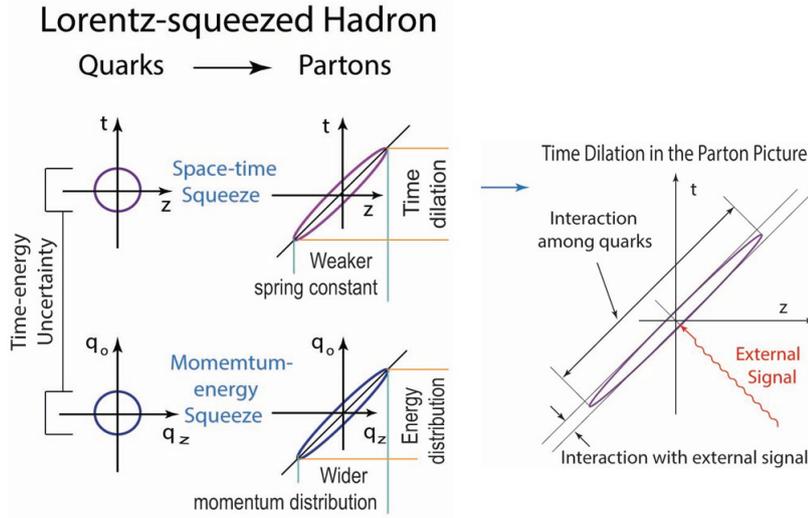


Figure 6.3: Lorentz-squeezed space-time and momentum-energy wave functions. As the hadron’s speed approaches that of light, both wave functions become concentrated along their respective positive light-cone axes. These light-cone concentrations lead to Feynman’s parton picture (Kim and Noz 1977, Kim 1989). The external signal, since it is moving in the direction opposite to the direction of the hadron, travels along the negative light-cone axis. Thus, the interaction time of this signal with the bound state is much shorter than the period of oscillation of the quarks inside the hadron. This effect is called Feynman’s time dilation (Feynman 1969, Kim and Noz 2005).

the wave function of Eq. (5.42) as

$$\psi_\eta(z, t) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[ \left( e^{2\eta}(z+t)^2 + e^{-2\eta}(z-t)^2 \right) \right] \right\}, \quad (6.1)$$

which becomes

$$\psi_\eta(z, t) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left[ (z^2 + t^2) \right] \right\}, \quad (6.2)$$

for  $\eta = 0$ .

## 6.2 Feynman’s parton picture

Let us go back to the two-body problem and discuss what happens to the wave function when the proton is Lorentz-boosted. For this system, we have discussed the Lorentz-squeeze problem in Sec. 5.2.

It is a widely accepted view that hadrons are quantum bound states of quarks having a localized probability distribution. As in all bound-state cases, this localization condition is responsible for the existence of discrete mass spectra. The most convincing evidence for this bound-state picture is the hadronic mass spectra (Feynman *et al.* 1971, Kim and Noz 1986).

However, this picture of bound states is applicable only to observers in the Lorentz frame in which the hadron is at rest. How would the hadrons appear to observers in other Lorentz frames?

In 1969, Feynman observed that a fast-moving hadron can be regarded as a collection of many "partons" whose properties appear to be quite different from those of the quarks (Feynman 1969, Kim and Noz 1986). For example, the number of quarks inside a static proton is three, while the number of partons in a rapidly moving proton appears to be infinite. The question then is how the proton looking like a bound state of quarks to one observer can appear so differently to an observer in a different Lorentz frame? Feynman made the following systematic observations.

- a. The picture is valid only for hadrons moving with velocity close to that of light.
- b. The interaction time between the quarks becomes dilated, and partons behave as free independent particles.
- c. The momentum distribution of partons becomes widespread as the hadron moves fast.
- d. The number of partons seems to be infinite or much larger than that of quarks.

Because the hadron is believed to be a bound state of two or three quarks, each of the above phenomena appears as a paradox, particularly b) and c) together. How can a free particle have a wide-spread momentum distribution?

In order to resolve this paradox, let us construct the momentum-energy wave function corresponding to Eq. (5.42). If the quarks have the four-momenta  $p_a$  and  $p_b$ , we can construct two independent four-momentum variables (Feynman *et al.* 1971)

$$P = p_a + p_b, \quad q = \sqrt{2}(p_a - p_b). \quad (6.3)$$

The four-momentum  $P$  is the total four-momentum and is thus the hadronic four-momentum while  $q$  measures the four-momentum separation between the quarks.

The resulting momentum-energy wave function is

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{4}\left[e^{-2\eta}(q_z + q_0)^2 + e^{2\eta}(q_z - q_0)^2\right]\right\}. \quad (6.4)$$

For large values of  $\eta$ , we can let  $q_0 = q_z$ , and the wave function becomes

$$\phi_\eta(q_z) = \left(\frac{1}{\pi}\right)^{1/4} \exp\left\{-\left[e^{-2\eta}(q_z)^2\right]\right\}. \quad (6.5)$$

Because we are using here the harmonic oscillator, the mathematical form of the above momentum-energy wave function is identical to that of the space-time wave function of Eq. (5.42). The Lorentz Squeeze properties of these wave functions are also the same. This aspect of the squeeze has been exhaustively discussed in the literature (Kim and Noz 1977, Kim 1989), and it is illustrated again in Fig. 6.3. The hadronic structure function calculated from this formalism is in reasonable agreement with the experimental data (Hussar 1981).

When the hadron is at rest with  $\eta = 0$ , both wave functions behave like those for the static bound state of quarks. As  $\eta$  increases, the wave functions become continuously squeezed until they become concentrated along their respective positive light-cone axes. Let us look at the z-axis projection of the space-time wave function. Indeed, the width of the quark distribution increases as the hadronic speed approaches that of the speed of light. The position of each quark appears widespread to the observer in the laboratory frame, and the quarks appear like free particles.

The momentum-energy wave function is just like the space-time wave function. The longitudinal momentum distribution becomes wide-spread as the hadronic speed approaches the velocity of light. This is in contradiction with our expectation from non-relativistic quantum mechanics that the width of the momentum distribution is inversely proportional to that of the position wave function. Our expectation is that if quarks are free, they must have a sharply defined momenta, not a wide-spread distribution.

However, according to our Lorentz-squeezed space-time and momentum-energy wave functions, the space-time width and the momentum-energy width increase in the same direction as the hadron is boosted. This is of course an effect of Lorentz covariance. This indeed leads to the resolution of one of the quark-parton puzzles (Kim and Noz 1977, 1986, Kim 1989).

Another puzzling problem in the parton picture is that partons appear as incoherent particles, while quarks are coherent when the hadron is at rest. Does this mean that the coherence is destroyed by the Lorentz boost (Kim 1998, 2004)? The answer is NO, and here is the resolution to this puzzle.

When the hadron is boosted, the hadronic matter becomes squeezed and becomes concentrated in the elliptic region along the positive light-cone axis. The length of the major axis becomes expanded by  $e^\eta$ , and the minor axis is contracted by  $e^{-\eta}$ .

This means that the interaction time of the quarks among themselves becomes dilated. Because the wave function becomes wide-spread, the distance between one end of the oscillator well and the other end increases. This effect, first noted by Feynman (Feynman 1969), is universally observed in high-energy hadronic experiments. The period of oscillation increases like  $e^\eta$ . On the other hand, the external signal, since it is moving in the direction opposite to the direction of the hadron, travels along the negative light-cone axis.

If the hadron contracts along the negative light-cone axis, the interaction time decreases by  $e^{-\eta}$ . The ratio of the interaction time to the oscillator period becomes  $e^{-2\eta}$ . The energy of each proton coming out of the LHC accelerator is  $13 \text{ TeV}$ . This leads to the ratio  $1.25 \times 10^{-9}$ . This is indeed a small number. The external signal is not able to sense the interaction of the quarks among themselves inside the hadron.

Indeed, the covariant harmonic oscillator formalism provides one Lorentz-covariant entity which produces the quark and parton models as two limiting cases as is indicated in Table 6.1.

### 6.3 Proton structure function

The quark distribution in momentum-energy space can be measured from the inelastic electron-proton scattering with one-photon exchange (Bjorken and Paschos 1969). The

Table 6.1: Further contents of Einstein's  $E = mc^2$ . The fourth row is added to Table 2.1. Indeed, the unified picture of the quark and parton models can be viewed as a further content of Einstein's energy-momentum relation (Kim and Noz 1986, Kim 1989).

|                               | Massive<br>Slow            | Lorentz<br>Covariance                    | Massless<br>Fast            |
|-------------------------------|----------------------------|--|-----------------------------|
| Energy-<br>Momentum           | $E = p^2/2m$               | Einstein's<br>$E = [p^2 + m^2]^{1/2}$    | $E = p$                     |
| Spin,Helicity<br>Gauge Trans. | $S_3$<br>$S_1, S_2$        | Wigner's<br>Little Group                 | Helicity<br>Gauge Trans.    |
| Hadron's<br>Constituents      | Gell-Mann's<br>Quark Model | Lorentz-covariant<br>Harmonic Oscillator | Feynman's<br>Parton Picture |

measured distribution is called the proton structure function. We are now interested in how close the Gaussian form of Eq.(6.4) is to the experimental world.

First of all, in the large- $\eta$  limit, the proton wave function is within the narrow elliptic region where  $q_z = q_0$ , and we are left with the wave function depending on only one variable. Thus, this one-variable wave function takes the form

$$\phi_\eta(q_z) = \left(\frac{1}{\pi}\right)^{1/4} \exp\left\{-\left[e^{-2\eta}(q_z)^2\right]\right\}. \quad (6.6)$$

According to Eq.(6.3), the

$$p_a = \sqrt{2}(P_z - 2p_a), \quad p_b = \sqrt{2}(P_z + 2p_a). \quad (6.7)$$

If we introduce the parameter

$$x = \frac{p_{az}}{P_z}, \quad (6.8)$$

This is the ratio of the quark momentum to the hadronic momentum. Indeed, this variable is used for measuring the parton distribution in high-energy laboratories.

It is then possible to write the Gaussian form of Eq.(6.6) in terms of this  $x$  variable, and the quark distribution can be written as

$$\rho(x) = \exp\left[-\gamma\left(x - \frac{1}{2}\right)^2\right], \quad (6.9)$$

where the constant  $\gamma$  is to be determined from the level separation from the hadronic mass spectra (Feynman *et al.* 1971). The variable  $x$  ranges from its minimum value of zero to the maximum value 1. This Gaussian form peaks at  $x = 1/2$ .

Before attempting to make a real contact with the experimental world, we have to face the fact that the proton is a bound state of three quarks. Within the harmonic oscillator regime, the three-body bound system can be separated into a regime of two independent oscillators. This problem was worked out in detail in the 1971 paper of Feynman *et al.*. Let us reproduce their calculation.

Let  $x_a, x_b, x_c$  represent the space-time coordinates for those quarks. If there is an oscillator force between two quarks, we are led the quadratic form

$$\left[ (x_a - x_b)^2 + (x_b - x_c)^2 + (x_c - x_a)^2 \right]. \quad (6.10)$$

In order to deal with this expression, Feynman *et al.* introduced the following three variables:

$$\begin{aligned} X &= \frac{x_a + x_b + x_c}{3}, \\ r &= \frac{x_a + x_b - 2x_c}{6}, \\ s &= \frac{x_b - x_a}{2}, \end{aligned} \quad (6.11)$$

and

$$\begin{aligned} x_a &= X - 2r, \\ x_b &= X + r - \sqrt{3}s, \\ x_c &= X + r + \sqrt{3}s. \end{aligned} \quad (6.12)$$

In terms of the  $r$  and  $s$  variables, the quadratic form becomes

$$18(r^2 + s^2), \quad (6.13)$$

and does not depend on the  $X$  variable, which specifies the space-time coordinate of the proton.

As for the momentum-energy four-vectors, let us call them  $p_a, p_b$ , and  $p_c$  for the quarks  $a, b, c$  respectively, and introduce the following variables. For the momentum-energy four-vectors, we can introduce the following three variables.

$$\begin{aligned} P &= p_a + p_b + p_c, \\ q &= p_a + p_b - 2p_c, \\ k &= \sqrt{3}(p_b - p_a). \end{aligned} \quad (6.14)$$

Then

$$\begin{aligned} p_a &= \frac{1}{3}P + \frac{1}{6}q - \frac{1}{2\sqrt{3}}k, \\ p_b &= \frac{1}{3}P + \frac{1}{6}q + \frac{1}{2\sqrt{3}}k, \\ p_c &= \frac{1}{3}P - \frac{1}{3}q. \end{aligned} \quad (6.15)$$

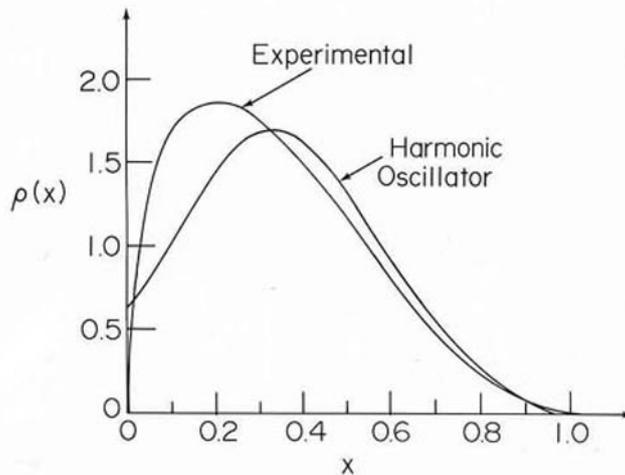


Figure 6.4: Parton distribution function compared with experimental data. The boosted oscillator has its peak at  $x = 1/3$ . This Gaussian form gives a reasonable agreement with experimental data for large values of  $x$ , but the disagreement is substantial for small values of  $x$ . This figure is from Paul Hussar's paper (Hussar 1981).

In terms of these variables we are led to consider the quadratic form of

$$18 (q^2 + k^2). \quad (6.16)$$

This form does not depend on the variable  $P$ , which measures the momentum and energy of the proton.

If the external signal interacts with quark  $c$ , its momentum depends only on the  $q$  variable, which can be written as

$$q = P - 3p_c. \quad (6.17)$$

We can then define the  $x$  variable as

$$x = \frac{p_{cz}}{P_z}. \quad (6.18)$$

Then the quark distribution should take the form

$$\rho(x) = \left( \frac{1}{\pi\gamma} \right)^{1/2} \exp \left[ -\gamma \left( 1 - \frac{1}{3} \right)^2 \right]. \quad (6.19)$$

The constant  $\gamma$  is to be determined from the hadronic mass spectra based on the oscillator model (Feynman *et al.* 1971). Figure 6.4 shows this Gaussian form, and its comparison with what we observe in the real world.

On the same figure, there is a curve derived from the distribution derived from the experimental data. This distribution is measured from inelastic electron-proton scattering (Bjorken and Paschos 1969). These two curves are somewhat different because the quarks do not interact with the incoming photon as a point particle.

The simplest model is to put all those effects into one additional quark in the oscillator system. This leads to the proton as a bound state of four quarks. The fourth quark is responsible for all those un-explained effects (Kim and Noz 1978). Another model is to

use the valon model (Hwa 1980, Hwa and Zahir 1981) which allows us to screen out all those non-point effects. Using this valon model, Hussar (1981) derived the experimental curve to be compared with the Gaussian form as shown in Fig. 6.4.

This graph may not be as accurate as we desire. However, the remarkable feature is that the Gaussian form was calculated from the proton at rest. So is the constant  $\gamma$ . It came from the level spacing in the hadronic mass spectra. It is remarkable that these two features manifest themselves for the proton whose speed is very close to that of the light.

There are many other models to deal with the problem of providing corrections to the parton distribution. QCD (quantum chromodynamics) is a case in point (Buras 1980). QCD can provide corrections to the distribution, but it does not produce the distribution from which to start. The covariant harmonic oscillator function provides this starting point.

It is like the case of quantum electrodynamics. QED was quite successful in producing the Lamb shift in the the hydrogen energy spectrum, but QED cannot produce the Rydberg energy levels to which the correction is made. The hydrogen energy levels are still obtained from the Schrödinger or Dirac equation with the localization condition on wave functions.

## 6.4 Proton form factor and Lorentz coherence

Let us now consider the elastic scattering of proton and electron with one photon exchange. If the proton is a point particle, the scattering cross section can be calculated from the one-photon exchange Feynman diagram. The calculation is straight-forward if the proton is a point particle. This process is called the Rutherford scattering, and the cross section becomes the same as the classical Coulomb scattering if the proton's recoil is negligible.

As the momentum transfer becomes substantial as indicated in Fig. 6.5 the cross section deviates from the that of the Rutherford scattering, as was observed first by Hofstadter and McCallister in 1955 (Hofstadter and McCallister 1955). Subsequently, it was observed that the cross section decreases as

$$\frac{1}{(\text{momentum transfer})^8} \tag{6.20}$$

This deviation comes from the fact that the proton is not a point particle and that the electric charge inside the proton is distributed with a finite radius. The portion of the scattering amplitude describing this distribution is called the proton form factor. The proton form factor should therefore decrease as

$$\frac{1}{(\text{momentum transfer})^4} \tag{6.21}$$

This behavior of decrease is known as the dipole cut-off in the literature. This dipole cut-off and possible deviations from it constitute one of the major branches of high-energy physics. There have been in the past some far-reaching theoretical models to deal with this problem (Frazer and Fulco 1960).

In this section, we are interested in approaching this problem using the harmonic oscillator formalism developed in Chapter 5. We shall show that the dipole cut-off is a

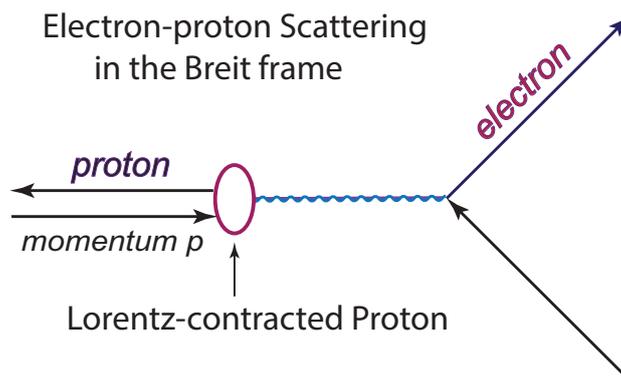


Figure 6.5: Breit frame. The incoming and outgoing protons move with equal magnitude of momentum in opposite directions (Kim and Noz 1986).

consequence of the coherence between the contraction of the proton wave function and the decrease in the wavelength of the incoming signal.

While the formalism of Chapter 5 is largely based on the papers written by Dirac and Wigner, it is interesting to note that the same harmonic oscillator functions can be derived from those authors who attempted to understand the proton form factor. These authors were not aware of the works of Dirac and Wigner. Let us briefly review what they did.

In 1953, Yukawa was interested in constructing harmonic oscillator wave functions that can be Lorentz-transformed (Yukawa 1953). His primary interest was in the mass spectrum produced by his Lorentz-invariant differential equation. However, at that time, his mass spectrum did not appear to have anything to do with the physical world.

After witnessing a non-zero charge radius of the proton observed by Hofstadter and McAllister (Hofstadter and McAllister 1955, Hofstadter 1956), Markov in 1956 considered using Yukawa's oscillator formalism for calculating the proton form factor (Markov 1956).

However, the constituent particles of the oscillator wave functions were not defined at that time. Shortly after the emergence of the quark model in 1964 (Gell-Mann 1964), Ginzburg and Man'ko (1985) considered relativistic harmonic oscillators for bound-states of quarks.

Even though they did not mention Yukawa's 1953 paper, Fujimura, Kobayashi, and Namiki used the quark model based on Yukawa's relativistic oscillator wave function, to calculate the proton form factor, and obtained the dipole cut-off (Fujimura *et al.* 1970). In the same year, Licht and Pagnamenta derived the same result using Lorentz-contracted oscillator wave functions. They used the Breit coordinate system in order to by-pass the time-separation variable appearing in the covariant formalism (Licht and Pagnamenta 1970, Kim and Noz 1973).

In 1971, Feynman, Kislinger, and Ravndal noted that the observed hadronic mass spectra can be explained in terms of the degeneracies of three-dimensional harmonic oscillators (Feynman *et al.* 1971), confirming the earlier suggestion made by Yukawa in 1953. They quoted the paper by Fujimura *et al.* (Fujimura *et al.* 1970), but they did not mention Yukawa's 1953 paper. This is the reason why Feynman *et al.* could not write down normalizable wave functions.

Let us go back to the formalism developed in Chapter 5. When considering the

scattering of one electron and one proton by exchanging one photon it is possible to choose the Lorentz frame in which the incoming and outgoing protons are moving in opposite directions with the same speed. Let us assume that the proton is moving along the  $z$  direction as indicated in Fig. 6.5, and let  $p$  be the magnitude of the momentum. Then the initial and final momentum-energy four-vectors are

$$(p, E) \quad \text{and} \quad (-p, E), \quad (6.22)$$

respectively, where  $E = \sqrt{1 + p^2}$ . The momentum transfer in this Breit frame is

$$(p, E) - (-p, E) = (2p, 0), \quad (6.23)$$

with zero energy component.

The proton form factor then becomes

$$F(p) = \int e^{2ipz} (\psi_\eta(z, t))^* \psi_{-\eta}(z, t) dz dt. \quad (6.24)$$

If we use the ground-state harmonic oscillator wave function, this integral becomes

$$\frac{1}{\pi} \int e^{2ipz} \exp \left\{ -\cosh(2\eta) (z^2 + t^2) \right\} dz dt. \quad (6.25)$$

The physics of  $\cosh(2\eta)$  in this expression was explained in Eq. (5.49).

In the Fourier integral of Eq. (6.25), the exponential function does not depend on the  $t$  variable. Thus, after the  $t$  integration, Eq. (6.25) becomes

$$F(p) = \frac{1}{\sqrt{\pi} \cosh(2\eta)} \int e^{2ipz} \exp \left\{ -z^2 \cosh(2\eta) \right\} dz. \quad (6.26)$$

If we complete this integral, the proton form factor becomes

$$F(p) = \frac{1}{\cosh(2\eta)} \exp \left\{ \frac{-p^2}{\cosh(2\eta)} \right\}. \quad (6.27)$$

If we use the expression of  $\cosh(2\eta)$  given in Eq. (5.49), this proton form factor becomes

$$F(p) = \frac{1}{1 + 2p^2} \exp \left( \frac{-p^2}{1 + 2p^2} \right), \quad (6.28)$$

which decreases as  $1/p^2$  for large values of  $p$ .

In order to illustrate the effect of the role of this Lorentz contraction in more detail, let us perform the integral of Eq. (6.26) without the contraction factor  $\cosh(2\eta)$ . This means that the wave function  $\psi_\eta(z, t)$  in the Eq. (6.24) is replaced by the Gaussian form  $\psi_0(z, t)$  of Eq. (6.2). With this non-squeezed wave function, the Fourier integral becomes

$$G(p) = \int e^{2ipz} (\psi_0(z, t))^* \psi_0(z, t) dz dt. \quad (6.29)$$

The result of this integral is

$$G(p) = \frac{1}{\sqrt{\pi}} \exp(-p^2). \quad (6.30)$$

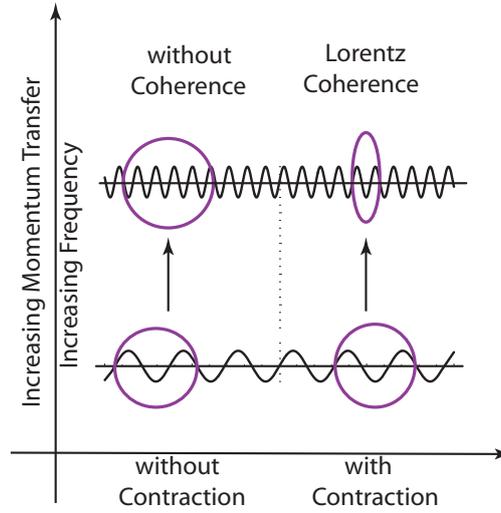


Figure 6.6: Coherence between the wavelength and the proton size. Referring back to Fig. 6.5, the proton sees the incoming photon. The wavelength of this photon becomes smaller for increasing momentum transfer. If the proton size remains unchanged, there is a rapid oscillation cutoff in the Fourier integral for the form factor leading to a Gaussian cutoff. However, if the proton size decreases coherently with the wavelength, there are no oscillation effects, leading to a polynomial decrease of the form factor (Kim and Noz 1986, 2011).

This leads to a Gaussian cutoff of the proton form factor. This does not happen in the real world, and the calculation of  $G(p)$  is for an illustrative purpose only.

Let us go back to the Fourier integrals of Eq. (6.24) and Eq. (6.29). The only difference is the  $\cosh(2\eta)$  factor in Eq. (6.24). This factor is in the normalization constant and comes from the integration over the  $t$  variable which does not affect the Fourier integral.

However, it causes the Gaussian width to shrink by  $1/\sqrt{2}p$  for large values of  $p$ . The wavelength of the sinusoidal factor is inversely proportional to the momentum  $2p$ . Thus, both the Gaussian width and the wavelength of the incoming signal shrink at the same rate of  $1/p$  as  $p$  becomes large. Without this coherence, the cutoff is Gaussian as noted in Eq. (6.30). The effect of this Lorentz coherence is illustrated in Fig. 6.6.

There still is a gap between  $F(p)$  of Eq. (6.28) and the real world. Before comparing this expression with the experimental data, we have to realize that there are three quarks inside the proton with two oscillator modes.

One of the modes goes through the Lorentz coherence process discussed in this section. The other mode goes through the contraction process given in Eq. (5.49). The net effect is

$$F_3(p) = \left( \frac{1}{1 + 2p^2} \right)^2 \exp \left( \frac{-p^2}{1 + 2p^2} \right). \quad (6.31)$$

This will lead to the desired dipole cut-off of  $(1/p^2)^2$ .

In addition, the effect of the quark spin should be addressed. There are also reports of deviations from the exact dipole cut-off. There have been attempts to study the proton form factors based on the four-dimensional rotation group with an imaginary time coordinate. There are also many papers based on the lattice QCD. A brief list of the

references to these efforts is given in (Kim and Noz 2011).

The purpose of this section was limited to studying in detail the role of Lorentz coherence in keeping the proton form factor from the steep Gaussian cutoff in the momentum transfer variable. The coherence problem is one of the primary issues of the current trend in physics.

## 6.5 Coherence in momentum-energy space

We are now interested in how Lorentz coherence manifests itself in momentum-energy space. We start with the Lorentz-squeezed wave function in momentum-energy space, which can be written as

$$\phi_{\eta}(q_z, q_0) = \frac{1}{2\pi} \int e^{-i(q_z z - q_0 t)} \psi_{\eta}(z, t) dt dz. \quad (6.32)$$

This is a Fourier transformation of the Lorentz-squeezed wave function of Eq. (6.1), where  $q_z$  and  $q_0$  are Fourier conjugate variables to  $z$  and  $t$  respectively. The result of this integral is

$$\phi_{\eta}(q_z, q_0) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[ e^{-2\eta} (q_z + q_0)^2 + e^{2\eta} (q_z - q_0)^2 \right] \right\}. \quad (6.33)$$

In terms of this momentum-energy wave function, the proton form factor of Eq. (6.24) can be written as

$$F(p) = \int \phi_{-\eta}^*(q_0, q_z - p) \phi_{\eta}(q_0, q_z + p) dq_0 dq_z. \quad (6.34)$$

The evaluation of this integral leads to the proton form factor  $F(p)$  given in Eq. (6.28).

In order to see the effect of the Lorentz coherence, let us look at two wave functions in Fig. 6.7. The integral is carried out over the  $q_z q_0$  plane. As the momentum  $p$  increases, the two wave functions become separated. Without the Lorentz squeeze, the wave functions do not overlap, and this leads to a sharp Gaussian cutoff as in the case of  $G(p)$  of Eq. (6.30).

On the other hand, the squeezed wave functions have an overlap as shown in Fig. 6.7, and this overlap becomes smaller as  $p$  increases. This leads to a slower polynomial cut-off (Kim and Noz 1986, 2011).

The discovery of the non-zero size of the proton opened a new era of physics (Hofstadter 1956). The proton is no longer a point particle. One way to measure its internal structure is to study the proton-electron scattering amplitude with one photon exchange, and its dependence on the momentum transfer. The deviation from the case with the point-particle proton is called the proton form factor.

On the experimental front, the dipole cut-off has been firmly established. Yes, there are also experimental results indicating deviations from this dipole behavior (Alkofer 2005, Matevosyan 2005). However, in the present section, no attempts have been made to review all the papers written on the corrections. From the theoretical point of view, those deviations are corrections from the basic dipole behavior.

While the study of the proton form factor is still a major subject in physics, it is gratifying to note that the proton's dipole cut-off comes from the coherence between the Lorentz contraction of the proton's longitudinal size and the decrease in the wavelength of the incoming signal.

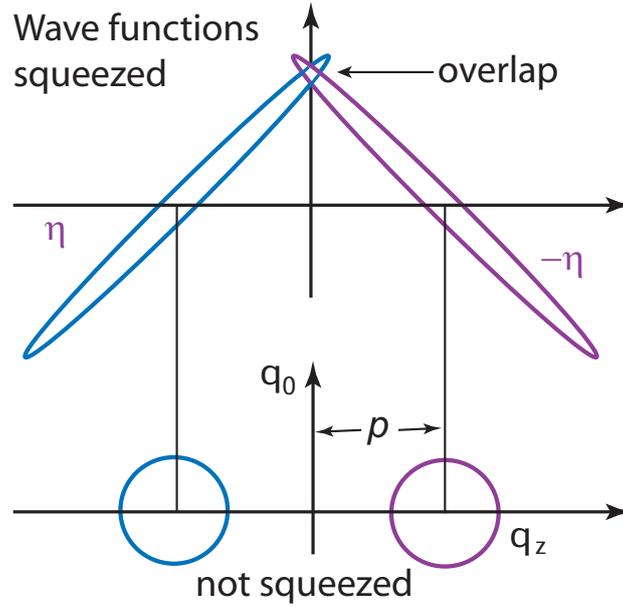


Figure 6.7: Lorentz coherence in the momentum-energy space. Both squeezed and non-squeezed wave functions are given. As  $p$  increases, the two wave functions in Eq. (6.34) become separated. Without the squeeze, there are no overlaps. This leads to a Gaussian cutoff. The squeezed wave functions maintain an overlap, leading to a slower polynomial cutoff (Kim and Noz 1986).

## 6.6 Hadronic temperature and boiling quarks

Harmonic oscillator wave functions are used for all branches of physics. The single-variable ground-state harmonic oscillator can be excited in the following three different ways.

1. Energy level excitations, with the energy eigenvalues  $\hbar\omega(n + 1/2)$ .
2. Coherent state excitations resulting in

$$|\alpha\rangle = e^{\alpha a^\dagger} = \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (6.35)$$

3. Thermal excitations resulting in the density matrix of the form

$$\rho_T(z, z') = \left(1 - e^{-\hbar\omega/kT}\right) \sum_k e^{-k\hbar\omega/kT} \phi_k(z) \phi_k^*(z'), \quad (6.36)$$

where  $\hbar\omega$  and  $k$  are the oscillator energy separation and Boltzmann's constant respectively. This form of the density matrix is well known (Landau and Lifshitz 1958, Davies and Davies 1975, Kim and Li 1989, Han *et al.* 1990).

We are now interested in the thermal excitation. If the temperature is measured in units of  $\hbar\omega/k$ , the density matrix of Eq. (6.36) can be written as

$$\rho_T(z, z') = \left(1 - e^{-1/T}\right) \sum_k e^{-1/T} \phi_k(z) \phi_k^*(z'), \quad (6.37)$$

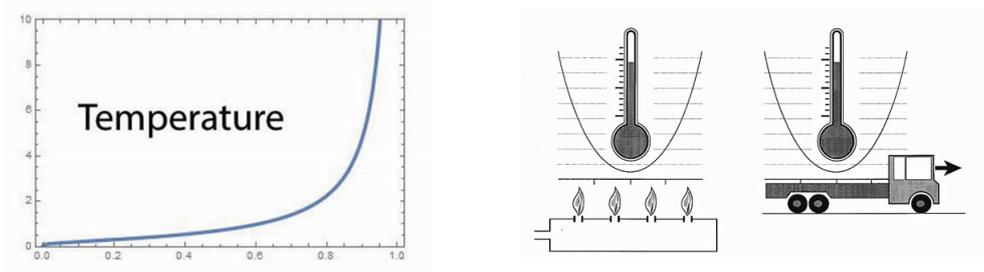


Figure 6.8: Hadronic temperature plotted against  $\beta$ . As the hadron gains in speed, the quarks inside become excited and this results in a rise in temperature. If the temperature is sufficiently high, those quarks start boiling and become partons (Kim and Noz 2014).

If we compare this expression with the density matrix of Eq. (5.59), we are led to

$$\tanh^2 \eta = \exp(-1/T), \quad (6.38)$$

and to

$$T = \frac{-1}{\ln(\tanh^2 \eta)} \quad (6.39)$$

The temperature can be calculated as a function of  $\tanh(\eta)$ , and this calculation is plotted in Fig. 6.8.

Earlier in Eq. (5.57), we noted that  $\tanh(\eta)$  is proportional to velocity of the hadron, and  $\tanh(\eta) = v/c$ . Thus, the oscillator becomes thermally excited as it moves, as is illustrated in Fig. 6.8.

Let us look at the velocity dependence of the temperature again. It is almost proportional to the velocity from  $\tanh(\eta) = 0$  to 0.7, and again from  $\tanh(\eta) = 0.9$  to 1 with different slopes.

While the physical motivation for this section was based on Feynman's time separation variable (Feynman *et al.* 1971) and his rest of the universe (Feynman 1972), we should note that many authors have discussed field theoretic approaches to derive the density matrix of Eq. (6.36). Among them are two-mode squeezed states of light (Yuen 1976, Yurke *et al.* 1986, Han *et al.* 1993, Kim and Noz 1991) and thermo-field-dynamics (Fetter and Walecka 1971, Ojima 1981, Umezawa *et al.* 1982, Mann and Revzen 1989).

The mathematics of two-mode squeezed states is the same as that for the covariant harmonic oscillator formalism discussed in Chapter 5 (Dirac 1963, Yurke 1986, Kim and Noz 1991, Han *et al.* 1993). Instead of the  $z$  and  $t$  coordinates, there are two measurable photons. If we choose not to observe one of them (Yurke and Potasek 1987, Ekert and Knight 1989), it belongs to Feynman's rest of the universe (Han *et al.* 1999)

Another remarkable feature of two-mode squeezed states of light is that its formalism is identical to that of thermo-field-dynamics. The temperature is related to the squeeze parameter in the two-mode case. It is therefore possible to define the temperature of a Lorentz-squeezed hadron within the framework of the covariant harmonic oscillator model.

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# Chapter 7

## Coupled oscillators and squeezed states of light

Let us start with the one-dimensional harmonic oscillator equation

$$\frac{1}{2} \left[ - \left( \frac{\partial}{\partial x} \right)^2 + x^2 \right] \chi_n(x) = \left( n + \frac{1}{2} \right) \chi_n(x), \quad (7.1)$$

whose solution takes the form

$$\chi_n(x) = \left[ \frac{1}{\sqrt{\pi} 2^n n!} \right]^{1/2} H_n(x) \exp \left( \frac{-x^2}{2} \right), \quad (7.2)$$

where  $H_n(x)$  is the Hermite polynomial of the  $n^{\text{th}}$  degree. The properties of this wave function are well known.

We can now consider two oscillators with coordinates  $x_1$  and  $x_2$  respectively. There are thus two differential equations with the  $x_1$  and  $x_2$  variables respectively. If we take the difference of these two equations, we obtain

$$\frac{1}{2} \left[ - \left( \frac{\partial}{\partial x_1} \right)^2 + \left( \frac{\partial}{\partial x_2} \right)^2 + x_1^2 - x_2^2 \right] \psi(x_1, x_2) = (n - n') \psi(x_1, x_2). \quad (7.3)$$

In terms of the light-cone variables defined in Sec. 4.3

$$u = \frac{x_1 + x_2}{\sqrt{2}}, \quad \text{and} \quad v = \frac{x_1 - x_2}{\sqrt{2}}, \quad (7.4)$$

the differential equation of Eq. (7.3) takes the form

$$\frac{1}{2} \left[ - \frac{\partial}{\partial u} \frac{\partial}{\partial v} + uv \right] \psi(u, v) = (n - n') \psi(u, v). \quad (7.5)$$

This equation is invariant under the transformation

$$u \rightarrow e^\eta u \quad \text{and} \quad v \rightarrow e^{-\eta} v. \quad (7.6)$$

Indeed, this was the property of the Lorentz-invariant differential equation which was the starting point of Chapter 5.

For the ground state, the solution takes the form

$$\psi_\eta(u, v) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left( e^{-2\eta} u^2 + e^{2\eta} v^2 \right) \right\}. \quad (7.7)$$

In terms of the  $x_1$   $x_2$  variables, it can be written as

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[ e^{-2\eta} (x_1 + x_2)^2 + e^{2\eta} (x_1 - x_2)^2 \right] \right\}. \quad (7.8)$$

We noted further that this Gaussian form can be expanded as

$$\psi_\eta(x_1, x_2) = \left( \frac{1}{\cosh \eta} \right) \sum_k (\tanh \eta)^k \chi_k(x_1) \chi_k(x_2). \quad (7.9)$$

In this chapter, we are interested in what happens if the two oscillators given in Eq. (7.3) are coupled with the equation

$$\frac{1}{2} \left[ - \left( \frac{\partial}{\partial x_1} \right)^2 - \left( \frac{\partial}{\partial x_2} \right)^2 + x_1^2 + x_2^2 + K(x_1 - x_2)^2 \right] f(x_1, x_2) = \lambda f(x_1, x_2), \quad (7.10)$$

where  $K$  measures the strength of the coupling.

## 7.1 Two coupled oscillators

The variables  $u$  and  $v$  of Eq. (7.4) are called the normal coordinates for two coupled oscillators. In terms of these variables, the differential equation of Eq. (7.10) becomes separable and can be written as

$$\frac{1}{2} \left[ - \left( \frac{\partial}{\partial u} \right)^2 - \left( \frac{\partial}{\partial v} \right)^2 + u^2 + e^{4\eta} v^2 \right] f(u, v) = \lambda f(u, v), \quad (7.11)$$

where

$$e^{4\eta} = 1 + K.$$

If we make the coordinate transformation

$$u \rightarrow e^\eta u, \quad v \rightarrow e^{-\eta} v, \quad (7.12)$$

the differential equation of Eq. (7.11) becomes

$$\frac{1}{2} \left[ \left( e^{2\eta} p_u^2 - e^{-2\eta} p_v^2 \right) + e^{-2\eta} u^2 + e^{2\eta} v^2 \right] f(u, v) = \lambda f(u, v), \quad (7.13)$$

with

$$p_u = -i \frac{\partial}{\partial u}, \quad \text{and} \quad p_v = -i \frac{\partial}{\partial v}. \quad (7.14)$$

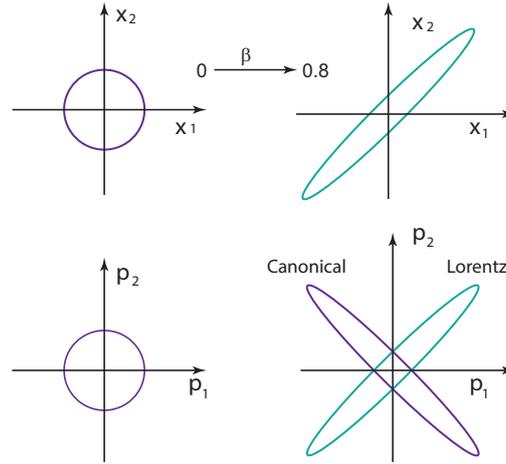


Figure 7.1: Lorentz transformations and canonical transformations. In the Lorentz transformations, the momentum wavefunction is squeezed in the same direction as the space wave function, while the canonical wave function is squeezed in the opposite direction to the space wave function.

According to Eq. (7.12), the transformation property of variables  $x_1, p_1$  and  $x_2, p_2$  can be written as

$$\begin{pmatrix} u' \\ v' \\ p'_u \\ p'_v \end{pmatrix} = \begin{pmatrix} e^\eta & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & e^{-\eta} & 0 \\ 0 & 0 & 0 & e^\eta \end{pmatrix} \begin{pmatrix} u \\ v \\ p_u \\ p_v \end{pmatrix}. \quad (7.15)$$

This is a canonical transformation, while the Lorentz transformation takes the form

$$\begin{pmatrix} u' \\ v' \\ p'_u \\ p'_v \end{pmatrix} = \begin{pmatrix} e^\eta & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & e^\eta & 0 \\ 0 & 0 & 0 & e^{-\eta} \end{pmatrix} \begin{pmatrix} u \\ v \\ p_u \\ p_v \end{pmatrix}. \quad (7.16)$$

as noted in Chapter 5. For both cases, the transformation in  $xy$  space is the same, but they are different in momentum space, as illustrated in Fig. 7.1.

For the ground state of this coupled system, the Gaussian form of the wave function becomes

$$f_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[ e^{-2\eta}(x_1 + x_2)^2 + e^{2\eta}(x_1 - x_2)^2 \right] \right\}, \quad (7.17)$$

as in the case of Eq. (7.8). According to Eq. (7.9), this Gaussian form can also be written as

$$f_\eta(x_1, x_2) = \left( \frac{1}{\cosh \eta} \right) \sum_k (\tanh \eta)^k \chi_k(x_1) \chi_k(x_2). \quad (7.18)$$

It is interesting to note that both the coupled oscillators and the covariant oscillator lead to the same Gaussian form of Eq. (7.17) with its series expansion. This series serves also as the starting formula for the two-photon coherent (Yuen 1976) as well as the Gaussian entangled state (Giedke *et al.* 2003, Dodd and Halliwell 2004, Braunstein and van Loock 2005, Adesso and Illuminati 2007, Paz and Roncaglia 2008, Chou *et al.* 2008, Xiang *et al.* 2011).

If  $\eta = 0$ , which means  $k = 0$  in Eq. (7.10), the Gaussian form of Eq. (7.17) becomes

$$f_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} (x_1^2 + x_2^2) \right\}, \quad (7.19)$$

and the series of Eq. (7.18) becomes disentangled to

$$f_0(x_1, x_2) = \chi_0(x_1)\chi_0(x_2). \quad (7.20)$$

## 7.2 Squeezed states of light

Let us go back to the complete set of oscillator functions given in Eq. (7.2) and introduce the operators  $a$  and  $a^\dagger$ , defined as

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right). \quad (7.21)$$

When these operators are applied to the wave functions, we obtain

$$a \chi_n(x) = \sqrt{n} \chi_{n-1}(x), \quad \text{and} \quad a^\dagger \chi_n = \sqrt{n+1} \chi_{n+1}(x). \quad (7.22)$$

Thus,  $\chi_n(x)$  can be used for the state of  $n$  photons, while  $a$  and  $a^\dagger$  can serve as the annihilation and creation operators respectively. If there are two kinds of photons, we can use

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{2}} \left( x_1 + \frac{\partial}{\partial x_1} \right), & a_1^\dagger &= \frac{1}{\sqrt{2}} \left( x_1 - \frac{\partial}{\partial x_1} \right), \\ a_2 &= \frac{1}{\sqrt{2}} \left( x_2 + \frac{\partial}{\partial x_2} \right), & a_2^\dagger &= \frac{1}{\sqrt{2}} \left( x_2 - \frac{\partial}{\partial x_2} \right). \end{aligned} \quad (7.23)$$

The single-photon coherent state takes the form

$$|\alpha\rangle = e^{-\alpha\alpha^*/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (7.24)$$

which can be expanded as

$$|\alpha\rangle = e^{-\alpha\alpha^*/2} \sum_n \frac{\alpha^n}{n!} (a^\dagger)^n |0\rangle = \left\{ e^{-\alpha\alpha^*/2} \right\} \exp \left\{ \alpha a^\dagger \right\} |0\rangle. \quad (7.25)$$

This aspect of the single-photon coherent state is well known. Here we are dealing with one kind of photon, namely with a given momentum and polarization. The state  $|n\rangle$  means there are  $n$  photons of the same kind, and  $|0\rangle$  is for the zero-photon vacuum state corresponding to the ground state oscillator wave function.

Let us next consider a state of two kinds of photons, and write  $|n_1, n_2\rangle$  as the state of  $n_1$  photons of the first kind, and  $n_2$  photons of the second kind (Yuen 1976). We are then led to the exponential form

$$|\beta\rangle = (1 - \beta^2)^{1/2} \exp \left\{ \beta a_1^\dagger a_2^\dagger \right\} |0, 0\rangle. \quad (7.26)$$

The Taylor expansion of this formula leads to

$$|\beta\rangle = (1 - \beta^2)^{1/2} \sum_k \beta^k |k, k\rangle, \quad (7.27)$$

which is the two-photon coherent state (Yuen 1976). In the language of harmonic oscillators, this formula is  $f_\eta(x_1, x_2)$  of Eq. (7.18) with

$$\beta = \tanh \eta. \quad (7.28)$$

In view of Fig. 7.1, it is quite appropriate to call this two-photon state the “squeezed state”.

The total energy of the state with  $n_1$  and  $n_2$  photons is clearly

$$\omega_1 n_1 + \omega_2 n_2, \quad (7.29)$$

where  $\omega_1$  and  $\omega_2$  are the frequencies of the first and second kinds respectively. Then the energy of the squeezed state of Eq. (7.8) is

$$(\omega_1 + \omega_2) (1 - \beta^2) \sum_k k \beta^{2k} = (\omega_1 + \omega_2) (1 - \beta^2) \frac{\beta}{2} \left( \frac{\partial}{\partial \beta} \sum_k \beta^{2k} \right). \quad (7.30)$$

which becomes

$$\frac{(\omega_1 + \omega_2) \beta^2}{(1 - \beta^2)} = (\omega_1 + \omega_2) (\sinh \eta)^2. \quad (7.31)$$

The energy is zero for the vacuum state with  $\eta = 0$ , but it increases as the system gets squeezed with increasing values of  $\eta$ .

Since the two-mode squeezed state and the covariant harmonic oscillators share the same set of mathematical formulas, it is possible to transmit physical interpretations from one to the other. For the two-mode squeezed state, both photons carry physical interpretations, while the interpretation is yet to be given to the time-separation variable in the covariant oscillator formalism. It is clear from Eq. (5.39) and Eq. (5.53) that the time-like excitations are like the second-photon states. What would happen if the second photon is not observed as is illustrated in Fig. 7.2?

This interesting problem was addressed by Yurke and Potasek (Yurke and Potasek 1987) and by Ekert and Knight (Ekert and Knight 1989). They used the density matrix formalism and integrated out the second-photon states, by using the same mathematics as in Sec. 5.7.

According to Yurke *et al.* (Yurke *et al.* 1986), it is possible to consider interferometers applicable to two-mode states. First of all, they picked the following three Hermitian operators

$$\begin{aligned} J_1 &= \frac{1}{2} (a_1^\dagger a_2 + a_2^\dagger a_1), \\ J_2 &= \frac{1}{2i} (a_1^\dagger a_2 - a_2^\dagger a_1), \\ J_3 &= \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \end{aligned} \quad (7.32)$$

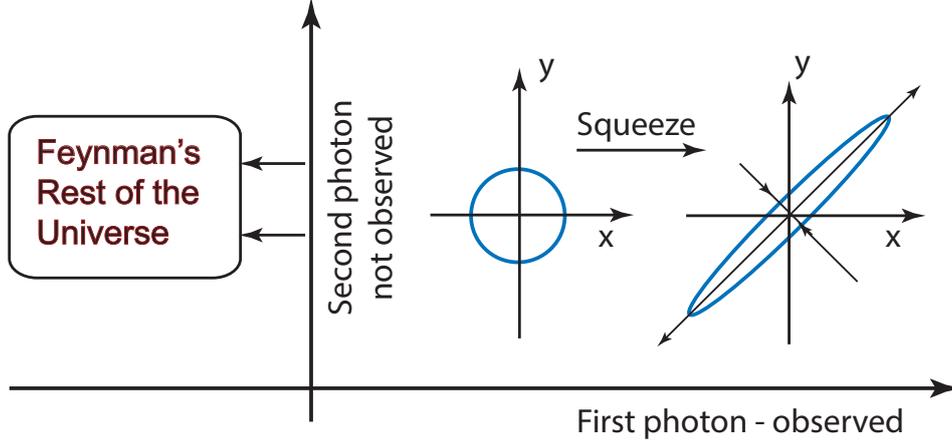


Figure 7.2: Feynman’s rest of the universe in the entangled system of two photons. If one of the photons is not observed, it is in the rest of the universe. This figure is identical with Fig. 5.8 except for the labeling of the coordinate axes.

in addition to the number operator

$$N = a_1^\dagger a_1 + a_2^\dagger a_2. \quad (7.33)$$

Since these operators satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad (7.34)$$

it is possible to study the symmetry property of the three-dimensional rotation group.

In addition, they wrote down another set of operators which could lead to effects on the two-mode states. They are

$$\begin{aligned} S_3 &= \frac{1}{2} (a_1^\dagger a_1 + a_2 a_2^\dagger), \\ Q_3 &= \frac{i}{2} (a_1^\dagger a_2^\dagger - a_1 a_2), \\ K_3 &= \frac{1}{2} (a_1^\dagger a_2^\dagger + a_1 a_2), \end{aligned} \quad (7.35)$$

where they satisfy the commutation relations

$$[K_3, Q_3] = -iS_3, \quad [Q_3, S_3] = iK_3, \quad [S_3, K_3] = iQ_3. \quad (7.36)$$

These relations are like those for the  $SU(1, 1)$  group or the Lorentz group  $O(2, 1)$ , applicable to two space dimensions and one time dimension.

The question is whether it is possible to combine the three-operators of Eq. (7.32) and those of Eq. (7.35).

### 7.3 $O(3,2)$ symmetry from Dirac’s coupled oscillators

In his 1963 paper, Dirac (Dirac 1963) started with the Schrödinger equation for two harmonic oscillators given in Eq. (7.11). We can now consider unitary transformations

applicable to the ground-state wave function of Eq. (7.19). Here, Dirac noted that those unitary transformations are generated by

$$\begin{aligned}
 L_1 &= \frac{1}{2} (a_1^\dagger a_2 + a_2^\dagger a_1), & L_2 &= \frac{1}{2i} (a_1^\dagger a_2 - a_2^\dagger a_1), \\
 L_3 &= \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), & S_3 &= \frac{1}{2} (a_1^\dagger a_1 + a_2 a_2^\dagger), \\
 K_1 &= -\frac{1}{4} (a_1^\dagger a_1^\dagger + a_1 a_1 - a_2^\dagger a_2^\dagger - a_2 a_2), \\
 K_2 &= \frac{i}{4} (a_1^\dagger a_1^\dagger - a_1 a_1 + a_2^\dagger a_2^\dagger - a_2 a_2), \\
 K_3 &= \frac{1}{2} (a_1^\dagger a_2^\dagger + a_1 a_2), \\
 Q_1 &= -\frac{i}{4} (a_1^\dagger a_1^\dagger - a_1 a_1 - a_2^\dagger a_2^\dagger + a_2 a_2), \\
 Q_2 &= -\frac{1}{4} (a_1^\dagger a_1^\dagger + a_1 a_1 + a_2^\dagger a_2^\dagger + a_2 a_2), \\
 Q_3 &= \frac{i}{2} (a_1^\dagger a_2^\dagger - a_1 a_2),
 \end{aligned} \tag{7.37}$$

where  $a^\dagger$  and  $a$  are the step-up and step-down operators applicable to harmonic oscillator wave functions. These operators satisfy the following set of commutation relations.

$$\begin{aligned}
 [L_i, L_j] &= i\epsilon_{ijk} L_k, & [L_i, K_j] &= i\epsilon_{ijk} K_k, & [L_i, Q_j] &= i\epsilon_{ijk} Q_k, \\
 [K_i, K_j] &= [Q_i, Q_j] = -i\epsilon_{ijk} L_k, & [L_i, S_3] &= 0, \\
 [K_i, Q_j] &= -i\delta_{ij} S_3, & [K_i, S_3] &= -iQ_i, & [Q_i, S_3] &= iK_i.
 \end{aligned} \tag{7.38}$$

Dirac then determined that these commutation relations constitute the Lie algebra for the  $O(3,2)$  de Sitter group with ten generators. This de Sitter group is the Lorentz group applicable to three space coordinates and two time coordinates. Let us use the notation  $(x, y, z, t, s)$ , with  $(x, y, z)$  as space coordinates and  $(t, s)$  as two time coordinates. Then the rotation around the  $z$  axis is generated by

$$L_3 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.39}$$

The generators  $L_1$  and  $L_2$  can be also be constructed. The  $K_3$  and  $Q_3$  generators will take the form

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}. \tag{7.40}$$

From these two matrices, the generators  $K_1, K_2, Q_1, Q_2$  can be constructed. The generator  $S_3$  can be written as

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}. \quad (7.41)$$

The last five-by-five matrix generates rotations in the two-dimensional space of  $(t, s)$ .

In his 1963 paper, Dirac states that the Lie algebra of Eq. (7.38) can serve as the four-dimensional symplectic group  $Sp(4)$ . In order to see this point, let us go to the Wigner phase-space picture of the coupled oscillators.

For a dynamical system consisting of two pairs of canonical variables  $x_1, p_1$  and  $x_2, p_2$ , we can use the coordinate variables defined as (Han *et al.* 1995)

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, p_1, x_2, p_2). \quad (7.42)$$

Then the four-by-four transformation matrix  $M$  applicable to this four-component vector is canonical (Abraham and Marsdan 1978, Goldstein 1980) if

$$MJ\tilde{M} = J, \quad (7.43)$$

with

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (7.44)$$

According to this form of the  $J$  matrix, the area of the phase space for the  $x_1$  and  $p_1$  variables remains invariant, and the story is the same for the phase space of  $x_2$  and  $p_2$ .

We can then write the generators of the  $Sp(4)$  group as

$$\begin{aligned} L_1 &= \frac{-1}{2} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, & L_2 &= \frac{i}{2} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \\ L_3 &= \frac{1}{2} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, & S_3 &= \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \end{aligned} \quad (7.45)$$

and

$$K_1 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad K_2 = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad K_3 = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

and

$$Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad Q_2 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad Q_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \quad (7.46)$$

These four-by-four matrices satisfy the commutation relations given in Eq. (7.38). Indeed, the de Sitter group  $O(3, 2)$  is locally isomorphic to the  $Sp(4)$  group. The remaining question is whether these ten matrices can serve as the fifteen Dirac matrices in the Majorana representation (Majorana 1932, Kim and Noz 2012). The answer is clearly No. How can ten matrices describe fifteen matrices? We should therefore add five more matrices. In order to address this question, we need Wigner functions.

## 7.4 O(3,3) symmetry from Dirac matrices

Since all the generators for the two coupled oscillator system can be written as four-by-four matrices with imaginary elements, it is convenient to work with Dirac matrices in the Majorana representation, where all the elements are imaginary (Majorana 1932, Itzykson and Zuber 1980, Lee 1995). In the Majorana representation, the four Dirac's  $\gamma$  matrices are

$$\begin{aligned}\gamma_1 &= i \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \\ \gamma_3 &= -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, & \gamma_0 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix},\end{aligned}\quad (7.47)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These  $\gamma$  matrices are transformed like four-vectors under Lorentz transformations. From these four matrices, we can construct one pseudo-scalar matrix

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad (7.48)$$

and a pseudo vector  $i\gamma_5\gamma_\mu$  consisting of

$$\begin{aligned}i\gamma_5\gamma_1 &= i \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, & i\gamma_5\gamma_2 &= -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \\ i\gamma_5\gamma_0 &= i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, & i\gamma_5\gamma_3 &= i \begin{pmatrix} -\sigma_3 & 0 \\ 0 & +\sigma_3 \end{pmatrix}.\end{aligned}\quad (7.49)$$

In addition, we can construct the tensor of the  $\gamma$  as

$$T_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \quad (7.50)$$

This antisymmetric tensor has six components. They are

$$i\gamma_0\gamma_1 = -i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad i\gamma_0\gamma_2 = -i \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad i\gamma_0\gamma_3 = -i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad (7.51)$$

and

$$i\gamma_1\gamma_2 = i \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad i\gamma_2\gamma_3 = -i \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad i\gamma_3\gamma_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \quad (7.52)$$

There are now fifteen linearly independent four-by-four matrices. They are all traceless, their components are imaginary (Lee 1995). We shall call these Dirac matrices in the Majorana representation.

As we saw in Sec. 7.3, Dirac (Dirac 1963) constructed a set of four-by-four matrices from two coupled harmonic oscillators, within the framework of quantum mechanics. He ended up with ten four-by-four matrices. It is of interest to compare his oscillator matrices and his fifteen Majorana matrices.

Unlike the case of the Schrödinger picture, it is possible to add five noncanonical generators to the list of generators given in Sec. 7.3. They are

$$\begin{aligned} S_1 &= +\frac{i}{2} \left\{ \left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) - \left( p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1} \right) \right\}, \\ S_2 &= -\frac{i}{2} \left\{ \left( x_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial x_1} \right) + \left( x_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial x_2} \right) \right\}, \end{aligned} \quad (7.53)$$

as well as three additional squeeze operators:

$$\begin{aligned} G_1 &= -\frac{i}{2} \left\{ \left( x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1} \right) + \left( p_1 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_1} \right) \right\}, \\ G_2 &= \frac{i}{2} \left\{ \left( x_1 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial x_1} \right) - \left( x_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial x_2} \right) \right\}, \\ G_3 &= -\frac{i}{2} \left\{ \left( x_1 \frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial p_1} \right) + \left( x_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial x_2} \right) \right\}. \end{aligned} \quad (7.54)$$

These five generators perform well-defined operations on the Wigner function. However, the question is whether these additional generators are acceptable in the present form of quantum mechanics.

In order to answer this question, let us note that the uncertainty principle in the phase-space picture of quantum mechanics is stated in terms of the minimum area in phase space for a given pair of conjugate variables. The minimum area is determined by Planck's constant. Thus we are allowed to expand phase space, but are not allowed to contract it. With this point in mind, let us go back to  $G_3$  of Eq. (7.54), which generates transformations which simultaneously expand one phase space and contract the other. Thus, the  $G_3$  generator is not acceptable in quantum mechanics even though it generates well-defined mathematical transformations of the Wigner function.

If the five generators of Eq. (7.53) and Eq. (7.54) are added to the ten generators given in Eq. (7.37) and Eq. (7.38), there are fifteen generators. They satisfy the following set of commutation relations.

$$\begin{aligned} [L_i, L_j] &= i\epsilon_{ijk}L_k, & [S_i, S_j] &= i\epsilon_{ijk}S_k, & [L_i, S_j] &= 0, \\ [L_i, K_j] &= i\epsilon_{ijk}K_k, & [L_i, Q_j] &= i\epsilon_{ijk}Q_k, & [L_i, G_j] &= i\epsilon_{ijk}G_k, \\ [K_i, K_j] &= [Q_i, Q_j] = [Q_i, Q_j] = -i\epsilon_{ijk}L_k, \\ [K_i, Q_j] &= -i\delta_{ij}S_3, & [Q_i, G_j] &= -i\delta_{ij}S_1, & [G_i, K_j] &= -i\delta_{ij}S_2, \\ [K_i, S_3] &= -iQ_i, & [Q_i, S_3] &= iK_i, & [G_i, S_3] &= 0, \\ [K_i, S_1] &= 0, & [Q_i, S_1] &= -iG_i, & [G_i, S_1] &= iQ_i, \\ [K_i, S_2] &= iG_i, & [Q_i, S_2] &= 0, & [G_i, S_2] &= -iK_i. \end{aligned} \quad (7.55)$$

As was shown previously (Kim and Noz 2012), this set of commutation relations serves as the Lie algebra for the group  $SL(4, r)$  and also for the  $O(3, 3)$  Lorentz group.

These fifteen four-by-four matrices are tabulated in Table 7.1. There are six anti-symmetric and nine symmetric matrices. These anti-symmetric matrices were divided into two sets of three rotation generators in the four-dimensional phase space. The nine symmetric matrices can be divided into three sets of three squeeze generators. However, this classification scheme is easier to understand in terms the group  $O(3, 3)$ .

Table 7.1:  $SL(4, r)$  and Dirac matrices. Two sets of rotation generators and three sets of boost generators. There are 15 generators.

|          | First component                      | Second component                     | Third component                      |
|----------|--------------------------------------|--------------------------------------|--------------------------------------|
| Rotation | $L_1 = \frac{-i}{2}\gamma_0$         | $L_2 = \frac{-i}{2}\gamma_5\gamma_0$ | $L_3 = \frac{-1}{2}\gamma_5$         |
| Rotation | $S_1 = \frac{i}{2}\gamma_2\gamma_3$  | $S_2 = \frac{i}{2}\gamma_1\gamma_2$  | $S_3 = \frac{i}{2}\gamma_3\gamma_1$  |
| Boost    | $K_1 = \frac{-i}{2}\gamma_5\gamma_1$ | $K_2 = \frac{1}{2}\gamma_1$          | $K_3 = \frac{i}{2}\gamma_0\gamma_1$  |
| Boost    | $Q_1 = \frac{i}{2}\gamma_5\gamma_3$  | $Q_2 = \frac{-1}{2}\gamma_3$         | $Q_3 = -\frac{i}{2}\gamma_0\gamma_3$ |
| Boost    | $G_1 = \frac{-i}{2}\gamma_5\gamma_2$ | $G_2 = \frac{1}{2}\gamma_2$          | $G_3 = \frac{i}{2}\gamma_0\gamma_2$  |

## 7.5 Non-canonical transformations in quantum mechanics

As we noted before, among the fifteen Dirac matrices, ten of them can be used for canonical transformations in classical mechanics, and thus in quantum mechanics. They play a special role in quantum optics (Yuen 1976, Yurke *et al.* 1986, Kim and Noz 1991, Han *et al.* 1993).

The remaining five of them can be interpreted if the change in phase space area is allowed. In quantum mechanics, the area can be increased, but it has a lower limit called Plank's constant. In classical mechanics, this constraint does not exist. The mathematical formalism given in this chapter allows us to study this aspect of the system of coupled oscillators.

Let us choose the following three matrices from those in Eqs. (7.45) and (7.46).

$$S_3 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad K_2 = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad Q_2 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad (7.56)$$

They satisfy the closed set of commutation relations:

$$[S_3, K_2] = iQ_2, \quad [S_3, Q_2] = -iK_2, \quad [K_2, Q_2] = -iS_3 \quad (7.57)$$

This is the Lie algebra for the  $Sp(2)$  group, This is the symmetry group applicable to the single-oscillator phase space (Kim and Noz 1991), with one rotation and two squeezes. These matrices generate the same transformation for the first and second oscillators.

We can choose three other sets with similar properties. They are

$$S_3 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad Q_1 = \frac{-i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad K_1 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad (7.58)$$

$$L_3 = \frac{1}{2} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad K_2 = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad K_1 = \frac{-i}{2} \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad (7.59)$$

and

$$L_3 = \frac{1}{2} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad Q_2 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \quad (7.60)$$

These matrices also satisfy the commutation relations given in Eq. (7.57). In this case, the squeeze transformations take opposite directions in the second phase space.

Since all these transformations are canonical, they leave the area of each phase space invariant. However, let us look at the non-canonical generator  $G_3$  of Eq. (7.54). It generates the transformation matrix of the form

$$\begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} \quad (7.61)$$

If  $\eta$  is positive, this matrix expands the first phase space while contracting the second. This contraction of the second phase space is allowed in classical mechanics, but it has a lower limit in quantum mechanics. This is illustrated in Fig. 7.3.

The expansion of the first phase space is exactly like the thermal expansion resulting from our failure to observe the second oscillator that belongs to the rest of the universe. If we expand the system of Dirac's ten oscillator matrices to the world of his fifteen Majorana matrices, we can expand and contract the first and second phase spaces without mixing them up. We can thus construct a model where the observed world and the rest of the universe remain separated. In the observable world, quantum mechanics remains valid with thermal excitations. In the rest of the universe, since the area of the phase space can decrease without lower limit, only classical mechanics is valid.

During the expansion/contraction process, the product of the areas of the two phase spaces remains constant. This may or may not be an extended interpretation of the uncertainty principle, but we choose not to speculate further on this issue.

Let us turn our attention to the fact that the groups  $SL(4, r)$  and  $Sp(4)$  are locally isomorphic to  $O(3, 3)$  and  $O(3, 2)$  respectively. This means that we can do quantum mechanics in one of the  $O(3, 2)$  subgroups of  $O(3, 3)$ , as Dirac noted in his 1963 paper. The remaining generators belong to Feynman's rest of the universe.

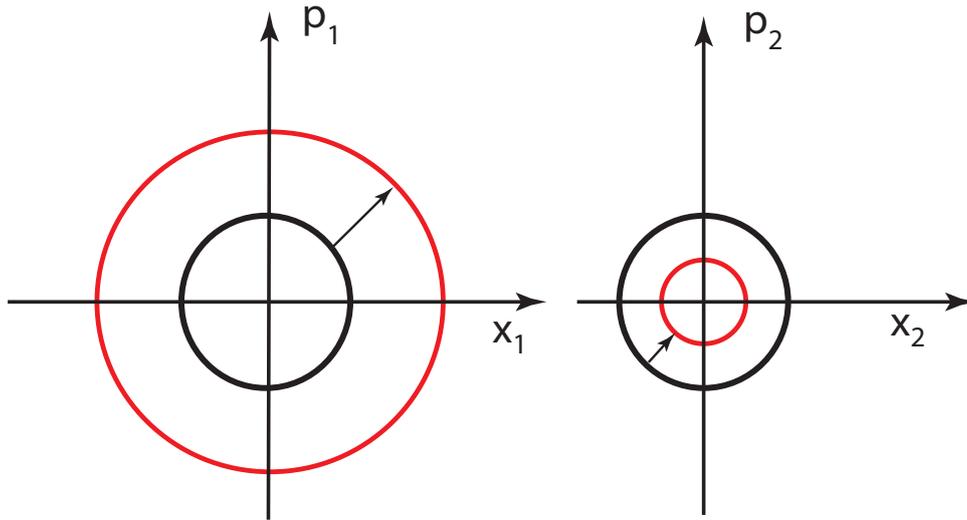


Figure 7.3: Expanding and contracting phase spaces. Canonical transformations leave the area of each phase space invariant. Non-canonical transformations can change them, yet the product of these two areas remains invariant (Kim and Noz 2012).

## 7.6 Entropy and the expanding Wigner phase space

We have seen how Feynman's rest of the universe increases the radius of the Wigner function. It is important to note that the entropy of the system also increases.

Let us go back to the density matrix. The standard way to measure this ignorance is to calculate the entropy defined as (von Neumann 1932, Landau and Lifshitz 1958, Fano 1957, Blum 1981, Kim and Wigner 1990, Kim and Noz 2014).

$$S = -Tr(\rho \ln(\rho)) \quad (7.62)$$

where  $S$  is measured in units of Boltzmann's constant. If we use the density matrix given in Eq. (5.53), the entropy becomes

$$S = 2 \left\{ \cosh^2 \left( \frac{\eta}{2} \right) \ln \left( \cosh \frac{\eta}{2} \right) - \sinh^2 \left( \frac{\eta}{2} \right) \ln \left( \sinh \frac{\eta}{2} \right) \right\} \quad (7.63)$$

In order to express this equation in terms of the temperature variable  $T$ , we write

$$\cosh \eta = \frac{1 + e^{-1/T}}{1 - e^{-1/T}} \quad (7.64)$$

which leads to

$$\cosh^2 \left( \frac{\eta}{2} \right) = \frac{1}{1 + e^{-1/T}}, \quad \sinh^2 \left( \frac{\eta}{2} \right) = \frac{e^{-1/T}}{1 + e^{-1/T}} \quad (7.65)$$

Then the entropy of Eq. (7.63) takes the form (Han *et al.* 1999, Kim and Wigner 1990)

$$S = \left( \frac{1}{T} \right) \left\{ \frac{1}{\exp(1/T) - 1} \right\} - \ln(1 - e^{-1/T}) \quad (7.66)$$

This familiar expression is for the entropy of an oscillator state in thermal equilibrium. Thus, for this oscillator system, we can relate our ignorance of Feynman's rest of the universe, measured by the coupling parameter  $\eta$ , to the temperature.

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# Chapter 8

## Lorentz group in ray optics

While the Lorentz group serves as the basic language for Einstein's special theory of relativity, it can also be considered to be the basic mathematical instrument in optical sciences, particularly in ray and polarization optics. In this chapter, the two-by-two beam transfer matrix, commonly called the  $ABCD$  matrix, is shown to be a two-by-two representation of the Lorentz group discussed in Sec. 1.2. It is thus possible to study this matrix in terms of the mathematical device developed for studying Wigner's little groups dictating internal space-time symmetries of particles in the Lorentz-covariant world discussed extensively in Chapter 3.

### 8.1 Group of $ABCD$ matrices

The Lorentz group for particle physics is generated by six matrices, however, the elements of the  $ABCD$  matrices are always real and unimodular (determinant = 1). Thus, this matrix can be generated by  $J_2, K_3$ , and  $K_1$  given in Chapter 1, which lead to two-by-two matrices with real elements. These generators satisfy the closed set of commutation relations

$$[J_2, K_1] = -iK_3, \quad [J_2, K_3] = iK_1, \quad [K_1, K_3] = -iJ_2, \quad (8.1)$$

where

$$J_2 = \frac{i}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (8.2)$$

They form the Lie algebra for the  $Sp(2)$  group (Georgieva and Kim 2001, Sánchez-Soto *et al.* 2012, Başkal and Kim 2013).

The representation of the  $Sp(2)$  group generated by these commutation relations consist of a rotation about the origin, a squeeze along the  $x$  direction and another squeeze at along axes rotated by  $45^\circ$  respectively. These generators are equivalent to another set of generators which form a representation of the  $Sp(2)$  group consisting of two shear transformations and a squeeze transformation (Başkal and Kim 2001, 2003). Therefore we shall use a rotation, a squeeze, and a shear matrix of the form (Başkal and Kim 2009):

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad (8.3)$$

and

$$T(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \quad (8.4)$$

The last matrix can also be viewed as an optical filter (Han, *et al.* 1999), or as a translation matrix (Başkal and Kim 2003).

The traces of these matrices are smaller than 2, equal to 2, and greater than 2, respectively. In Chapter 4, we discussed how to make transitions from one to another.

Since the  $ABCD$  matrix has real elements and a determinant of one, it has three independent parameters. The elements of the  $ABCD$  matrix are determined by optical materials and how they are arranged. The purpose of this chapter is to explore the mathematical properties of the  $ABCD$  matrix which can address more fundamental issues in physics.

## 8.2 Equi-diagonalization of the $ABCD$ matrix

In dealing with matrices, it is a routine process to diagonalize them. However, it is not always possible. The triangular matrix in Eq. (8.4) cannot be diagonalized. On the other hand, the two diagonal elements are equal.

The  $ABCD$  matrix with inputs from optical instruments is not always equi-diagonal. Thus, we have to equi-diagonalize the matrix. Let us start from the matrix

$$[ABCD] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (8.5)$$

where  $A$  and  $D$  are not necessarily equal to each other. Then the transformation

$$B(\eta) [ABCD] B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \quad (8.6)$$

will lead to the equi-diagonal form

$$\begin{pmatrix} \sqrt{AD} & B \\ C & \sqrt{AD} \end{pmatrix} \quad (8.7)$$

with

$$e^\eta = \sqrt{D/A}.$$

The diagonal matrix  $B(\eta)$  is given in Eq. (8.3). This form of equi-diagonalization will be useful in camera optics discussed in Sec. 8.6.

Although the transformation of Eq. (8.6) is an unimodular (determinant-preserving) transformation, it is not a similarity transformation. Let us next consider a rotation of the  $ABCD$  matrix

$$R(\theta) [ABCD] R(-\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (8.8)$$

which leads to the matrix

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}, \quad (8.9)$$

with

$$\begin{aligned} A' &= \frac{1}{2}[A(1 + \cos \theta) + D(1 - \cos \theta) - (C + B) \sin \theta], \\ D' &= \frac{1}{2}[A(1 - \cos \theta) + D(1 + \cos \theta) + (C + B) \sin \theta]. \end{aligned} \quad (8.10)$$

If these two diagonal elements are to be equal,

$$\tan \theta = \frac{A - D}{A + B}. \quad (8.11)$$

We now have two different ways of transforming the  $ABCD$  matrix into a equi-diagonal form. Is it possible to make these into one mathematical transformation? Let  $M$  be an arbitrary element of the  $Sp(2)$  group. We can define the ‘‘Hermitian transformation’’ of the  $ABCD$  matrix as

$$M [ABCD] M^\dagger, \quad (8.12)$$

where  $M^\dagger$  is the Hermitian conjugate of  $M$ .

The Hermitian transformation of Eq. (8.12) is like the Lorentz transformation on the four-vector discussed in Chapter 3. If  $M$  is Hermitian, and is anti-symmetric, its Hermitian conjugate is its inverse. Thus the transformation is a similarity transformation. If  $M$  is symmetric, its Hermitian conjugate is not its inverse. Thus the transformation is not a similarity transformation.

The rotation matrix of Eq. (8.3) is anti-symmetric and its Hermitian conjugate is its inverse. Thus, the transformation of Eq. (8.12) with the rotation matrix is a similarity transformation. The squeeze matrix, such as  $B(\eta)$  is symmetric, and it is invariant under Hermitian conjugation. The Hermitian transformation of Eq. (8.12) with the squeeze matrix is not a similarity transformation.

Once the  $ABCD$  matrix is equi-diagonalized, it can be brought to one of the forms (Başkal and Kim 2009, 2010)

$$\begin{aligned} & \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \\ & \begin{pmatrix} 1 & -e^\eta \gamma \\ 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} \cosh(\lambda/2) & -e^\eta \sinh(\lambda/2) \\ -e^{-\eta} \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}. \end{aligned} \quad (8.13)$$

We shall use the notation  $W$  for these three equi-diagonal matrices. It is then possible to use all the mathematical instruments developed for the two-by-two representation of Wigner’s little group in Chapter 3. The expressions for these two-by-two matrices are given in Table 3.2.

These three matrices, like the ones defined in Eqs. (8.3) and (8.4) form different classes with different traces. They are smaller than, equal to, and greater than 2 respectively. However, it is possible to make continuations from one to another through tangential continuity as noted in Chapter 4.

### 8.3 Decomposition of the ABCD matrix

The equi-diagonal matrices of Eq. (8.13) can now be written as

$$B(\eta) W [B(\eta)]^{-1}, \quad (8.14)$$

where  $W$  is one of the three single-parameter matrices:

$$\begin{aligned} R(\theta) &= \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \\ T(\gamma) &= \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \\ S(\lambda) &= \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}. \end{aligned} \quad (8.15)$$

These matrices are given in Table 3.1, but they perform different physical operations. The transformation of Eq. (8.14) is not a Hermitian transformation. It is a similarity transformation.

We have seen above that the two-by-two  $ABCD$  matrix can be written as a similarity transformation of one of the three possible  $W$  matrices given in Eq. (8.15). Thus, if the  $ABCD$  matrix is equi-diagonalized by a rotation, it can be written as a similarity transformation

$$[ABCD] = R(\sigma)B(\eta) W B(-\eta)R(-\sigma) = [R(\sigma)B(\eta)] W [R(\sigma)B(\eta)]^{-1}. \quad (8.16)$$

Thus, the repeated application of the  $ABCD$  matrix becomes

$$[ABCD]^N = [R(\sigma)B(\eta)] W^N [R(\sigma)B(\eta)]^{-1}. \quad (8.17)$$

There is another form of decomposition known as the Bargmann decomposition discussed in Sec. 3.3. This procedure combines all three different classes of Eq. (8.13) into one analytic expression.

$$W = R(\alpha)S(-2\chi)R(\alpha), \quad (8.18)$$

where the forms of the rotation matrix  $R$  and the squeeze matrix  $S$  are given in Eq. (8.15). If we carry out the matrix multiplication, the  $W$  matrix becomes

$$\begin{pmatrix} (\cosh \chi) \cos \alpha & -\sinh \chi - (\cosh \chi) \sin \alpha \\ -\sinh \chi + (\cosh \chi) \sin \alpha & (\cosh \chi) \cos \alpha \end{pmatrix}. \quad (8.19)$$

This matrix also has two independent variables  $\alpha$  and  $\chi$ . We can write these parameters in terms of the  $\eta$ ,  $\theta$ , and  $\gamma$  for the matrices given in Eq. (8.13) by comparing the matrix elements.

If the off-diagonal elements have different signs, with  $(\cosh \chi) \sin \alpha > \sinh \chi$ , the diagonal elements become

$$\cos(\theta/2) = (\cosh \chi) \cos \alpha. \quad (8.20)$$

The off-diagonal elements lead to

$$e^{2\eta} = \frac{(\cosh \chi) \sin \alpha + \sinh \chi}{(\cosh \chi) \sin \alpha - \sinh \chi} \quad (8.21)$$

If the off-diagonal elements have the same sign, the diagonal elements become

$$\cosh(\lambda/2) = (\cosh \chi) \cos \alpha, \quad (8.22)$$

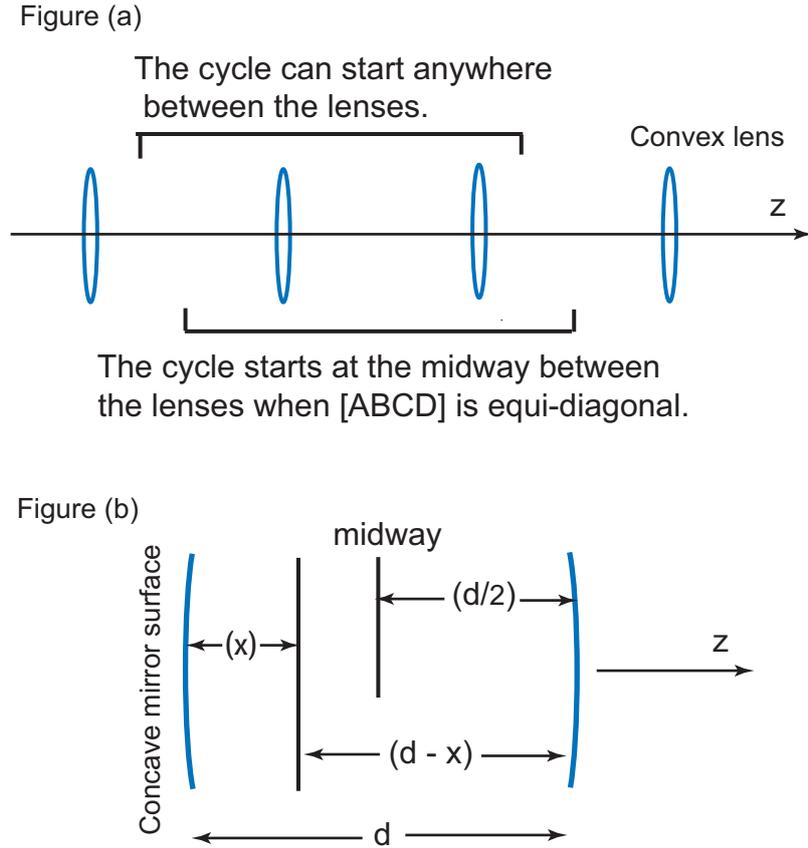


Figure 8.1: Optical rays in the laser cavity. fig.(a) illustrates that reflected beams are like those going through a series of convex lenses. fig.(b) describes how the beams are reflected from concave mirror surfaces (Başkal and Kim 2013).

with

$$e^{2\eta} = \frac{(\cosh \chi) \sin \alpha + \sinh \chi}{\sinh \chi - (\cosh \chi) \sin \alpha}. \quad (8.23)$$

If the lower left element of Eq. (8.19) vanishes, the matrix becomes

$$\begin{pmatrix} 1 & -2 \sinh \chi \\ 0 & 1 \end{pmatrix}. \quad (8.24)$$

The matrix of Eq. (8.19) is useful when we make transition from one form to another among the three matrices given in Eq. (8.13). We will need this procedure when we discuss multi-layer optics.

## 8.4 Laser cavities

As the first example of the periodic system, let us consider the laser cavity consisting of two identical concave mirrors separated by a distance  $d$  as shown in Fig. 8.1. Then the  $ABCD$  matrix for a round trip of one beam is

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad (8.25)$$

where the matrices

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad (8.26)$$

are the mirror and translation matrices respectively. The parameters  $R$  and  $d$  are the radius of the mirror and the mirror separation respectively. This form is quite familiar to us from the laser literature (Yariv 1975, Haus 1984, Hawkes *et al.* 1995).

The question then is what happens when this process is repeated. We are thus led to the question of whether the chain of matrices in Eq. (8.25) can be brought to an equi-diagonal form and eventually to a form of the Wigner decomposition. For this purpose, we rewrite the matrix of Eq. (8.25) as

$$\begin{aligned} & \begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}^2 \\ & \times \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (8.27)$$

In this way, we translate the system by  $-d/2$  using a translation matrix given in Eq. (8.26), and write the  $ABCD$  matrix of Eq. (8.25) as

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \left[ \begin{pmatrix} 1 - d/R & d - d^2/2R \\ -2/R & 1 - d/R \end{pmatrix} \right]^2 \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}. \quad (8.28)$$

We are thus led to concentrate on the matrix in the middle

$$\begin{pmatrix} 1 - d/R & d - d^2/2R \\ -2/R & 1 - d/R \end{pmatrix}, \quad (8.29)$$

which can be written as

$$\begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \begin{pmatrix} 1 - d/R & 1 - d/2R \\ -2d/R & 1 - d/R \end{pmatrix} \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}. \quad (8.30)$$

It is then possible to decompose the  $ABCD$  matrix into

$$E C^2 E^{-1}, \quad (8.31)$$

with

$$\begin{aligned} C &= \begin{pmatrix} 1 - d/R & 1 - d/2R \\ -2d/R & 1 - d/R \end{pmatrix}, \\ E &= \begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix}. \end{aligned} \quad (8.32)$$

The  $C$  matrix now contains only dimensionless numbers, and it can be written as

$$C = \begin{pmatrix} \cos(\gamma/2) & e^\eta \sin(\gamma/2) \\ -e^{-\eta} \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix}, \quad (8.33)$$

with

$$\begin{aligned} \cos(\gamma/2) &= 1 - \frac{d}{R}, \\ e^\eta &= \sqrt{\frac{2R - d}{4d}}. \end{aligned} \quad (8.34)$$

Here both  $d$  and  $R$  are positive, and the restriction on them is that  $d$  be smaller than  $2R$ . This is the stability condition frequently mentioned in the literature (Haus 1984, Hawkes *et al.* 1995).

Thus, the equi-diagonal matrix is  $C^2$ , and takes the form

$$\begin{pmatrix} \cos(\gamma) & e^\eta \sin(\gamma) \\ -e^{-\eta} \sin(\gamma) & \cos(\gamma) \end{pmatrix}, \quad (8.35)$$

which is

$$B(\eta)R(2\gamma)B(-\eta) \quad (8.36)$$

The  $ABCD$  matrix takes the form

$$[ABCD] = [E \ B(\eta)] R(2\chi) [E \ B(\eta)]^{-1}. \quad (8.37)$$

Thus, the repeated application becomes

$$[ABCD]^N = [E \ B(\eta)] R(2N\chi) [E \ B(\eta)]^{-1}. \quad (8.38)$$

## 8.5 Multilayer optics

We consider an optical beam going through a periodic medium with two different refractive indexes. If the beam traveling in the first medium hits the second medium, it is partially transmitted and partially reflected. In order to maintain the continuity of the Poynting vector, we define the electric fields as

$$\begin{aligned} E_1^{(\pm)} &= \frac{1}{\sqrt{n_1}} \exp(\pm ik_1 z - \omega t), \\ E_2^{(\pm)} &= \frac{1}{\sqrt{n_2}} \exp(\pm ik_2 z - \omega t) \end{aligned} \quad (8.39)$$

for the optical beams in the first and second media respectively. The superscript (+) and (−) are for the incoming and reflected rays respectively.

These two optical rays are related by the two-by-two  $ABCD$  matrix, according to

$$\begin{pmatrix} E_2^{(+)} \\ E_2^{(-)} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_1^{(+)} \\ E_1^{(-)} \end{pmatrix}. \quad (8.40)$$

Of course the elements of this matrix are determined by transmission coefficients as well as the phase shifts the beams experience while going through the media (Azzam and Bashara 1977, Georgieva and Kim 2001).

When the beam goes through the first medium to the second, we may use the boundary matrix given by Azzam and Bashara (Azzam and Bashara 1977) and by Monzón and Sánchez-Soto (Monzón and Sánchez-Soto 1999, 2000, Dragoman 2010). In terms of the refractive indexes  $n_1$  and  $n_2$ ,

$$S(\sigma) = \begin{pmatrix} \cosh(\sigma/2) & \sinh(\sigma/2) \\ \sinh(\sigma/2) & \cosh(\sigma/2) \end{pmatrix}, \quad (8.41)$$

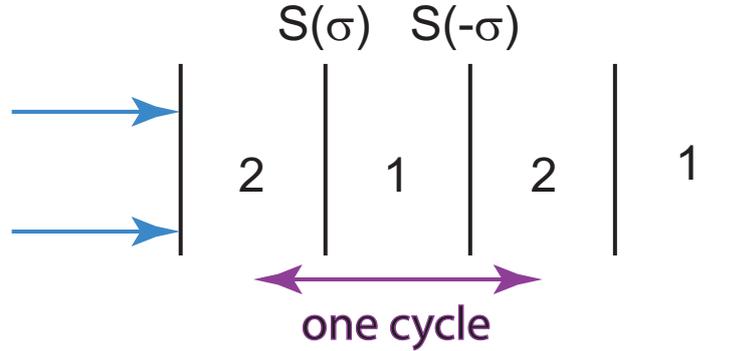


Figure 8.2: Multi-layers. The cycle can start in such a way that the ABCD matrix becomes equi-diagonal (Georgieva and Kim 2001, 2003).

where we can write the  $\sigma$  parameter as

$$\cosh\left(\frac{\sigma}{2}\right) = \frac{n_1 + n_2}{2\sqrt{n_1 n_2}}, \quad \sinh\left(\frac{\sigma}{2}\right) = \frac{n_1 - n_2}{2\sqrt{n_1 n_2}}. \quad (8.42)$$

The boundary matrix for the beam going from the second medium should be  $S(-\sigma)$  as seen in Fig. 8.2.

In addition, we have to consider the phase shifts through which the beams have to travel. When the beam goes through the first media, we can use the phase-shift matrix

$$P(\delta_1) = \begin{pmatrix} e^{-i\delta_1/2} & 0 \\ 0 & e^{i\delta_1/2} \end{pmatrix}, \quad (8.43)$$

and a similar expression for  $P(\delta_2)$  for the second medium. The phase shift  $\delta$  is determined by the wave number and thickness of the medium.

We are thus led to consider one complete cycle starting from the midpoint of the second medium, and write

$$P(\delta_2/2) S(\sigma) P(\delta_1) S(-\sigma) P(\delta_2/2). \quad (8.44)$$

If multiplied into one matrix, is this matrix equi-diagonal to accept the Wigner and Bargmann decompositions? Another question is whether the matrices in the above expression can be converted into matrices with real elements.

In order to answer the second question, let us consider the similarity transformation

$$C_1 Z(\delta) D(\sigma) C_1^{-1}, \quad (8.45)$$

with

$$C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (8.46)$$

This transformation leads to

$$R(\delta) S(\sigma), \quad (8.47)$$

where

$$R(\delta) = \begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{pmatrix}. \quad (8.48)$$

This notation is consistent with the rotation matrices used in Sec. 8.3.

Let us make another similarity transformation with

$$C_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (8.49)$$

This changes  $S(\sigma)$  into  $B(\sigma)$  without changing  $R(\delta)$ , where

$$B(\sigma) = \begin{pmatrix} e^{\sigma/2} & 0 \\ 0 & e^{-\sigma/2} \end{pmatrix}, \quad (8.50)$$

again consistent with the  $B(\eta)$  matrix used in Sec. 8.3.

Thus the net similarity transformation matrix is (Georgieva and Kim 2001)

$$C = C_2 C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}, \quad (8.51)$$

with

$$C^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} & -e^{i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix}. \quad (8.52)$$

If we apply this similarity transformation to the long matrix chain of Eq. (8.44), it becomes another chain

$$M = R(\delta_2/2) B(\sigma) R(\delta_1) B(-\sigma) R(\delta_2/2), \quad (8.53)$$

where all the matrices are real.

Let us now address the main question of whether this matrix chain can be brought to one equi-diagonal matrix. We note first that the three middle matrices can be written in a familiar form

$$\begin{aligned} M &= B(\sigma) R(\delta_1) B(-\sigma) \\ &= \begin{pmatrix} \cos(\delta_1/2) & -e^\sigma \sin(\delta_1/2) \\ e^{-\sigma} \sin(\delta_1/2) & \cos(\delta_1/2) \end{pmatrix}. \end{aligned} \quad (8.54)$$

However, due to the rotation matrix  $R(\delta_2/2)$  at the beginning and at the end of Eq. (8.53), it is not clear whether the entire chain can be written as a similarity transformation.

In order to resolve this issue, let us write Eq. (8.54) as a Bargmann decomposition

$$R(\alpha) S(-2\chi) R(\alpha), \quad (8.55)$$

with its explicit expression given in Eq. (8.19). The parameters  $\alpha$  and  $\chi$  are related to  $\sigma$  and  $\delta_1$  by

$$\begin{aligned} \cos(\delta_1/2) &= (\cosh \chi) \cos \alpha, \\ e^{2\eta} &= \frac{(\cosh \chi) \sin \alpha + \sinh \chi}{(\cosh \chi) \sin \alpha - \sinh \chi}. \end{aligned} \quad (8.56)$$

It is now clear that the entire chain of Eq. (8.44) can be written as another Bargmann decomposition

$$M = R(\alpha + \delta_2/2)S(-2\chi)R(\alpha + \delta_2/2). \quad (8.57)$$

Finally, this expression can be converted to a Wigner decomposition (Georgieva and Kim 2003)

$$M = B(\eta)R(\theta)B(-\eta), \quad (8.58)$$

with

$$\begin{aligned} \cos(\theta/2) &= (\cosh \chi) \cos(\alpha + \delta_2/2), \\ e^{2\eta} &= \frac{(\cosh \chi) \sin(\alpha + \delta_2/2) + \sinh \chi}{(\cosh \chi) \sin(\alpha + \delta_2/2) - \sinh \chi}. \end{aligned} \quad (8.59)$$

The decomposition of Eq. (8.58) allows us to deal with the periodic system of multilayers. For repeated application of  $M$ , we can now write

$$M^N = B(\eta)R(N\theta)B(-\eta). \quad (8.60)$$

## 8.6 Camera optics

The basic optical arrangement for the camera consists of a lens with focal length  $f$  and the propagation of the ray by an amount  $d$  (Saleh and Teich 2007). The lens matrix is given by

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \quad (8.61)$$

and a translation of the ray is expressed by the matrix

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}. \quad (8.62)$$

If the object and the image are  $d_1$  and  $d_2$  distances away from the lens respectively, the system is described by

$$\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}. \quad (8.63)$$

The multiplication of these matrices leads to the camera matrix of the form

$$\begin{pmatrix} 1 - d_2/f & d_1 + d_2 - d_1d_2/f \\ -1/f & 1 - d_1/f \end{pmatrix}. \quad (8.64)$$

The image becomes focused when the upper right element of this matrix vanishes (Başkal and Kim 2003), i.e.,

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}. \quad (8.65)$$

For the camera optics, both  $d_1$  and  $d_2$  are longer than  $f$ . It is then more convenient to deal with the negative of the matrix given in Eq. (8.64) with positive diagonal elements. In addition, let us use the dimensionless variables

$$x_1 = d_1/f, \quad x_2 = d_2/f. \quad (8.66)$$

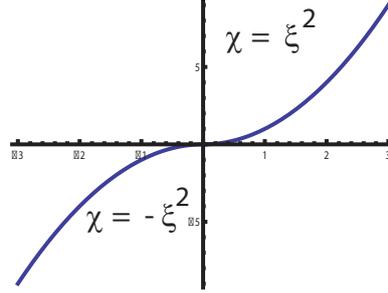


Figure 8.3: Tangential continuity in the camera focusing. The vertical axis  $\chi$  of this graph is the upper-right element of the camera matrix and is  $-\sin^2(\theta/2)$  for  $\xi < 0$ , and  $\sinh^2(\lambda/2)$  for  $\xi > 0$  (Başkal and Kim 2014).

Then the camera matrix becomes

$$\begin{pmatrix} x_2 - 1 & (x_1 - 1)(x_2 - 1) - 1 \\ 1 & x_1 - 1 \end{pmatrix}, \quad (8.67)$$

with the focal condition

$$\frac{1}{x_1} + \frac{1}{x_2} = 1. \quad (8.68)$$

If we wish to study the formula of the camera matrix Eq. (8.67) as a representation of the Lorentz group, the first step is to obtain its equi-diagonal form. However, if the focal condition of Eq. 8.68 is to be preserved, the off-diagonal elements should remain invariant. We thus have to resort to the Hermitian transformation given in Eq. (8.6). For this purpose, we can use the transformation matrix

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \quad (8.69)$$

with

$$e^\eta = \sqrt{\frac{1-x_2}{1-x_1}}. \quad (8.70)$$

The diagonal elements become

$$\sqrt{(1-x_1)(1-x_2)}. \quad (8.71)$$

If the diagonal elements are smaller than one, the camera matrix should take the form

$$\begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \quad (8.72)$$

with

$$e^{-\eta} \sin(\theta/2) = 1. \quad (8.73)$$

Thus the matrix becomes

$$\begin{pmatrix} \cos(\theta/2) & -\sin^2(\theta/2) \\ 1 & \cos(\theta/2) \end{pmatrix}. \quad (8.74)$$

If the diagonal elements are greater than one, the camera matrix becomes

$$\begin{pmatrix} \cosh(\lambda/2) & e^\eta \sinh(\lambda/2) \\ e^{-\eta} \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix} \quad (8.75)$$

with

$$e^{-\eta} \sinh(\lambda/2) = 1, \quad (8.76)$$

leading to

$$\begin{pmatrix} \cosh(\lambda/2) & \sinh^2(\lambda/2) \\ 1 & \cosh(\lambda/2) \end{pmatrix}. \quad (8.77)$$

Thus, the focusing process is the transition from

$$\begin{pmatrix} 1 - \xi^2/2 & -\xi^2 \\ 1 & 1 - \xi^2/2 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 + \xi^2/2 & \xi^2 \\ 1 & 1 + \xi^2/2 \end{pmatrix}, \quad (8.78)$$

via  $\xi = 0$ . This transition is illustrated in Fig. 8.3. Indeed, this is the tangential continuity discussed extensively in Chapter 4.

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# Chapter 9

## Polarization optics

In studying polarized light propagating along the  $z$  direction, the traditional approach is to consider the  $x$  and  $y$  components of the electric fields. Their amplitude ratio and phase difference determine the state of polarization. Thus, we can change the polarization either by adjusting the amplitudes, by changing the relative phases, or both. For convenience, we call the optical device which changes amplitudes an “attenuator” and the device which changes the relative phase a “phase shifter.”

It is thus possible to describe polarization of light as two-component column vectors, and two-by-two matrices applicable to those vectors. The two-by-two representation of the Lorentz group was discussed in Sec. 1.2. We will examine the polarization of light using the methods developed in the early chapters of this book.

The mathematics of the Lorentz group was originally developed for understanding Einstein’s special relativity. However, it is interesting to note that the same set of mathematical tools can be used for studying polarization optics. The Jones vectors, Mueller matrices, and Stokes parameters all constitute appropriate representations of the Lorentz group (Opatrny and Perina 1993, Tudor 2015, Han *et al.* 1997, Ben-Aryeh 2005, Red’kov 2011, Bařkal and Kim 2013, Franssens 2015).

### 9.1 Jones vectors

The traditional language for studying the two-component light vector is the Jones-matrix formalism which is covered in standard optics textbooks (Hecht 2002, Saleh and Teich 2007). In this formalism, the two transverse components of the electric fields are combined into one column matrix with the exponential form for the sinusoidal function

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A \exp \{i(kz - \omega t + \phi_1)\} \\ B \exp \{i(kz - \omega t + \phi_2)\} \end{pmatrix}. \quad (9.1)$$

This column matrix is called the Jones vector (Jones 1941, 1947) .

In the existing textbooks (Hecht 2002), the Jones-matrix formalism (Hecht 1970) starts with the projection operator

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (9.2)$$

applicable to the Jones vector of Eq. (9.1). This operator keeps the  $x$  component and completely eliminates the  $y$ -component of the electric field.

This is an oversimplification of the real world when the attenuation factor in the  $y$  direction is greater than that along the  $x$  direction. We shall replace this projection operator by an attenuation matrix which is closer to the real world.

In this section, we replace the projection operator of Eq. (9.2) by a squeeze matrix. There are two transverse directions which are perpendicular to each other. The absorption coefficient in one transverse direction could be different from the coefficient along the other direction. This can be described by the two-by-two matrix (Opatrny and Perina 1993, Han *et al.* 1997, Ben-Aryeh 2005, Tudor 2010, 2015, Bařkal and Kim 2013, Franssens 2015)

$$\begin{pmatrix} e^{-\mu_1} & 0 \\ 0 & e^{-\mu_2} \end{pmatrix} = e^{-(\mu_1+\mu_2)/2} \begin{pmatrix} e^{\mu/2} & 0 \\ 0 & e^{-\mu/2} \end{pmatrix} \quad (9.3)$$

with  $\mu = \mu_2 - \mu_1$ . Let us look at the projection operator of Eq. (9.2). Physically, it means that the absorption coefficient along the  $y$  direction is much larger than along the  $x$  direction. The absorption matrix in Eq. (9.3) becomes the projection matrix if  $\mu_1$  is very close to zero and  $\mu_2$  becomes infinitely large. The projection operator of Eq. (9.2) is therefore a special case of the above attenuation matrix.

The attenuation matrix of Eq. (9.3) tells us that the electric fields are attenuated at two different rates. The exponential factor  $e^{-(\mu_1+\mu_2)/2}$  reduces both components at the same rate and does not affect the state of polarization. The effect of polarization is solely determined by the squeeze matrix

$$B(\mu) = \begin{pmatrix} e^{\mu/2} & 0 \\ 0 & e^{-\mu/2} \end{pmatrix}. \quad (9.4)$$

The diagonal matrix of this type served many different purposes in earlier chapters of this book. It is a key element of the two-by-two representation of the Lorentz group.

Another basic element is the optical filter with different values of the index of refraction along the two orthogonal directions. The effect on this filter can be written as

$$\begin{pmatrix} e^{-i\delta_1} & 0 \\ 0 & e^{-i\delta_2} \end{pmatrix} = e^{-i(\delta_1+\delta_2)/2} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}, \quad (9.5)$$

with  $\delta = \delta_1 - \delta_2$ . In measurement processes, the overall phase factor  $e^{-i(\delta_1+\delta_2)/2}$  cannot be detected, and can therefore be deleted. The polarization effect of the filter is solely determined by the matrix

$$Z(\delta) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}, \quad (9.6)$$

which leads to a phase difference of  $\delta$  between the  $x$  and  $y$  components. The mathematical expression for this matrix is given in Eq. (9.4). It has a different physical meaning in the symmetry of the Lorentz group.

The polarization axes are not always the  $x$  and  $y$  axes. For this reason, we need the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (9.7)$$

to describe the rotation around the  $z$  axis.

The traditional Jones-matrix formalism consists of systematic combinations of the three components given in Eq. (9.2), Eq. (9.6), and Eq. (9.7). However, in this chapter, we shall use the squeeze matrix of Eq. (9.4) instead of the projection operator of Eq. (9.2). Then they become the starters of the two-by-two representation of the Lorentz group.

## 9.2 Squeeze and phase shift

The effect of the phase shift matrix  $Z(\delta)$  of Eq. (9.6) on the Jones vector is well known, but the effect of the squeeze matrix of Eq. (9.4) is not addressed adequately in the literature. Let us discuss the combined effect of these two matrices. First of all both are diagonal and they commute with each other.

The effect of the squeeze matrix on the Jones vector is straight-forward. If we apply the squeeze matrix of Eq. (9.4) to the Jones vector, the net result is

$$\begin{pmatrix} e^{\mu/2} & 0 \\ 0 & e^{-\mu/2} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} e^{\mu/2} E_x \\ e^{-\mu/2} E_y \end{pmatrix}. \quad (9.8)$$

This squeeze transformation expands one amplitude, while contracting the other so that the product of the amplitude remains invariant. This squeeze transformation is illustrated in Fig. 9.1.

In order to illustrate phase shifts, we start with the Jones vector of the form

$$\begin{pmatrix} \exp(ikz) \\ \exp[i(kz - \pi/2)] \end{pmatrix}, \quad (9.9)$$

whose real part is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(kz) \\ \sin(kz) \end{pmatrix}, \quad (9.10)$$

which corresponds to a circular polarization with

$$x^2 + y^2 = 1. \quad (9.11)$$

If we apply the phase shift matrix, the resulting vector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(kz + \delta/2) \\ \sin(kz - \delta/2) \end{pmatrix}, \quad (9.12)$$

which can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(kz - \pi/4 + \alpha) \\ \cos(kz - \pi/4 - \alpha) \end{pmatrix}, \quad (9.13)$$

with

$$\alpha = \frac{\delta}{2} + \frac{\pi}{4}. \quad (9.14)$$

Then

$$\begin{aligned} x + y &= 2(\cos \alpha) \cos(kz - \pi/4), \\ x - y &= 2(\sin \alpha) \sin(kz - \pi/4), \end{aligned} \quad (9.15)$$

and

$$\frac{(x + y)^2}{4(\cos \alpha)^2} + \frac{(x - y)^2}{4(\sin \alpha)^2} = 1. \quad (9.16)$$

This is an elliptic polarization.

The squeeze operation of Eq. (9.4) is relatively simple. It changes the amplitudes, and it commutes with the phase shift matrix. Thus, the combined effect could be illustrated in Fig. 9.1.

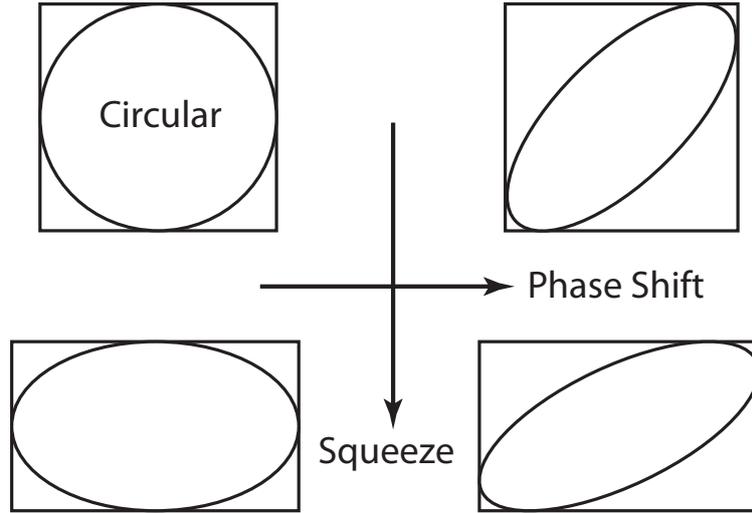


Figure 9.1: Squeeze and Phase Shift. Both squeeze and phase shifts are elliptic deformations, but they are done differently (Başkal and Kim 2013).

### 9.3 Rotation of the polarization axes

If the polarization coordinate is the same as the  $xy$  coordinate where the electric field components take the form of Eq. (9.1), the above attenuator is directly applicable to the column matrix of Eq. (9.1). If the polarization coordinate is rotated by an angle  $\theta/2$ , or by the matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (9.17)$$

the phase shifter takes the form

$$Z(\theta, \delta) = R(\theta)Z(\delta)R(-\theta) \quad (9.18)$$

$$= \begin{pmatrix} \cos(\delta/2) + i \sin(\delta/2) \cos \theta & i \sin(\delta/2) \sin \theta \\ i \sin(\delta/2) \sin \theta & \cos(\delta/2) - i \sin(\delta/2) \cos \theta \end{pmatrix}. \quad (9.19)$$

If the polarization coordinate system is rotated by  $45^\circ$ , the phase shifter matrix becomes

$$Q(\delta) = \begin{pmatrix} \cos(\delta/2) & i \sin(\delta/2) \\ i \sin(\delta/2) & \cos(\delta/2) \end{pmatrix}. \quad (9.20)$$

In order to illustrate what this matrix does to the polarized beams, let us start with the circularly polarized wave

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(ikz - i\omega t)}, \quad (9.21)$$

whose real part is

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(kz - \omega t) \\ \sin(kz - \omega t) \end{pmatrix}. \quad (9.22)$$

This leads to the familiar equation for the circle

$$X^2 + Y^2 = 1. \quad (9.23)$$

If the phase shifter of Eq. (9.20) is applied to the above Jones vector, the result is

$$\begin{pmatrix} [\cos(\delta/2) + \sin(\delta/2)] \cos(kz - \omega t) \\ i[\sin(\delta/2) - \cos(\delta/2)] \sin(kz - \omega t) \end{pmatrix} \quad (9.24)$$

with

$$\begin{aligned} \cos(\delta/2) &= \cos([\delta/2 + \pi/4] - \pi/4), \\ \sin(\delta/2) &= \cos([\delta/2 + \pi/4] + \pi/4). \end{aligned} \quad (9.25)$$

Thus,

$$\begin{aligned} \cos(\delta/2) + \sin(\delta/2) &= \sqrt{2} \cos(\delta/2 + \pi/4), \\ \cos(\delta/2) - \sin(\delta/2) &= \sqrt{2} \sin(\delta/2 + \pi/4). \end{aligned} \quad (9.26)$$

After the phase shift, the Jones vector becomes

$$\begin{pmatrix} [\sqrt{2} \cos \alpha] \cos(kz - \omega t) \\ [\sqrt{2} \sin \alpha] \sin(kz - \omega t) \end{pmatrix}, \quad (9.27)$$

with

$$\alpha = \frac{\delta}{2} + \frac{\pi}{4}. \quad (9.28)$$

The the  $x$  and  $y$  components will satisfy the equation

$$\frac{X^2}{(\sqrt{2} \cos \alpha)^2} + \frac{Y^2}{(\sqrt{2} \sin \alpha)^2} = 1. \quad (9.29)$$

This is an elliptic polarization. These steps are illustrated in Fig. 9.2.

Let us next consider rotations of the squeeze matrix.

$$B(\theta, \mu) = R(\theta)B(\mu)R(-\theta), \quad (9.30)$$

which leads to

$$B(\theta, \mu) = \begin{pmatrix} \cosh(\mu/2) + \sinh(\mu/2) \cos \theta & \sinh(\mu/2) \sin \theta \\ \sinh(\mu/2) \sin \theta & \cosh(\mu/2) - \sinh(\mu/2) \cos \theta \end{pmatrix}. \quad (9.31)$$

We are familiar with this squeeze operation from Sec. 8.3. This changes the amplitudes. If the squeeze angle become  $45^\circ$ , we use the notation  $S(\mu)$  for this special angle, and

$$S(\mu) = \begin{pmatrix} \cosh(\mu/2) & \sinh(\mu/2) \\ \sinh(\mu/2) & \cosh(\mu/2) \end{pmatrix}. \quad (9.32)$$

The question is what happens if two squeeze transformations are made in two different directions. Would the result be another squeeze? The answer is No. The result is another squeeze matrix followed by a rotation, which can be written as (Başkal and Kim 2005)

$$B(\theta, \lambda)B(0, \eta) = B(\phi, \xi)R(\omega), \quad (9.33)$$

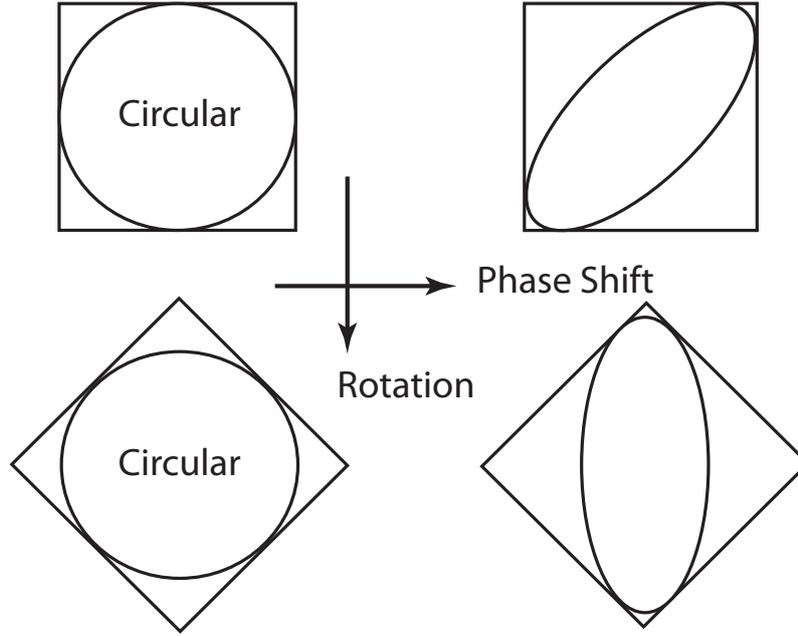


Figure 9.2: Phase shift and rotation. They are rotated by  $45^\circ$ . (Başkal and Kim 2013).

where

$$\begin{aligned} \cosh \xi &= \cosh \eta \cosh \lambda + \sinh \eta \sinh \lambda \cos \theta, \\ \tan \phi &= \frac{\sin \theta [\sinh \lambda + \tanh \eta (\cosh \lambda - 1) \cos \theta]}{\sinh \lambda \cos \theta + \tanh \eta [1 + (\cosh \lambda - 1) \cos^2 \theta]}, \\ \tan \omega &= \frac{2(\sin \theta) [\sinh \lambda \sinh \eta + C_- \cos \theta]}{C_+ + C_- \cos(2\theta) + 2 \sinh \lambda \sinh \eta \cos \theta}, \end{aligned} \quad (9.34)$$

with

$$C_{\pm} = (\cosh \lambda \pm 1)(\cosh \eta \pm 1). \quad (9.35)$$

Indeed, we write Eq. (9.33) as

$$R(\omega) = B(\phi, -\xi) B(\theta, \lambda) B(0, \eta), \quad (9.36)$$

three squeeze transformations lead to one rotation.

We have done this calculation using the kinematics of Lorentz transformations. On the other hand, it does not appear possible to do experiments using high-energy particles. However, it is gratifying to note that this experiment is possible in polarization optics.

If the angle  $\theta$  is  $90^\circ$ , the calculation becomes simpler, and

$$B(\lambda)B(\eta) = B(\phi, \xi)R(\omega), \quad (9.37)$$

where

$$\cosh \xi = \cosh \eta \cosh \lambda,$$

$$\begin{aligned}\tan \phi &= \frac{\sinh \lambda}{\tanh \eta}, \\ \tan \omega &= \frac{\sinh \lambda \sinh \eta}{\cosh \eta + \cosh \lambda}.\end{aligned}\tag{9.38}$$

## 9.4 Optical activities

For convenience, let us change the parameters  $\theta$  and  $\mu$  as

$$\theta = 2\alpha z, \quad \mu = 2\beta z.\tag{9.39}$$

Then the  $R(\theta)$  matrix can be written as

$$R(\alpha z) = \begin{pmatrix} \cos(\alpha z) & -\sin(\alpha z) \\ \sin(\alpha z) & \cos(\alpha z) \end{pmatrix},\tag{9.40}$$

and the rotation angle increases as the beam propagates along the  $z$  direction. This version of optical activity is well known.

In addition, we can consider the squeeze operation

$$S(-\beta z) = \begin{pmatrix} \cosh(\beta z) & -\sinh(\beta z) \\ -\sinh(\beta z) & \cosh(\beta z) \end{pmatrix}.\tag{9.41}$$

Here the squeeze parameter increases as the beam moves. The negative sign for  $\beta$  is for convenience.

If this squeeze is followed by the rotation of Eq. (9.40), the net effect is

$$\begin{pmatrix} \cosh(\beta z) & -\sinh(\beta z) \\ -\sinh(\beta z) & \cosh(\beta z) \end{pmatrix}, \begin{pmatrix} \cos(\alpha z) & -\sin(\alpha z) \\ \sin(\alpha z) & \cos(\alpha z) \end{pmatrix}\tag{9.42}$$

where  $z$  is in a macroscopic scale, perhaps measured in centimeters. However, this is not an accurate description of the optical process.

When this happens in a microscopic scale of  $z/N$ , it can become accumulated into the macroscopic scale of  $z$  after the  $N$  repetitions, where  $N$  is a very large number. We are thus led to the transformation matrix of the form (Kim 2010)

$$M(\alpha, \beta, z) = [S(-\beta z/N)R(\alpha z/N)]^N.\tag{9.43}$$

In the limit of large  $N$ , this quantity becomes

$$\left[ \begin{pmatrix} 1 & -\beta z/N \\ -\beta z/N & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha z/N \\ \alpha z/N & 1 \end{pmatrix} \right]^N.\tag{9.44}$$

Since  $\alpha z/N$  and  $\beta z/N$  are very small,

$$M(\alpha, \beta, z) = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -(\alpha + \beta) \\ (\alpha - \beta) & 0 \end{pmatrix} \frac{z}{N} \right]^N.\tag{9.45}$$

For large  $N$ , we can write this matrix as

$$M(\alpha, \beta, z) = \exp(Gz),\tag{9.46}$$

with

$$G = \begin{pmatrix} 0 & -(\alpha + \beta) \\ (\alpha - \beta) & 0 \end{pmatrix}. \quad (9.47)$$

We can compute this matrix using the procedure developed in Sec. 8.3. If  $\alpha$  is greater than  $\beta$ ,  $G$  becomes

$$G = \alpha' \begin{pmatrix} 0 & \exp(\eta) \\ \exp(-\eta) & 0 \end{pmatrix}, \quad (9.48)$$

with

$$\begin{aligned} \alpha' &= \sqrt{\alpha^2 - \beta^2}, \\ \exp(\eta) &= \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}, \end{aligned} \quad (9.49)$$

and the  $M$  matrix of Eq. (9.46) take the form

$$\begin{pmatrix} \cos(\alpha'z) & -e^\eta \sin(\alpha'z) \\ e^{-\eta} \sin(\alpha'z) & \cos(\alpha'z) \end{pmatrix}. \quad (9.50)$$

If  $\beta$  is greater than  $\alpha$ , the off-diagonal elements have the same sign. We can then write  $G$  as

$$G = -\beta' \begin{pmatrix} 0 & \exp(\eta) \\ \exp(-\eta) & 0 \end{pmatrix}, \quad (9.51)$$

with

$$\begin{aligned} \beta' &= \sqrt{\beta^2 - \alpha^2}, \\ \exp(\eta) &= \sqrt{\frac{\beta + \alpha}{\beta - \alpha}}, \end{aligned} \quad (9.52)$$

and the  $M$  matrix of Eq. (9.46) becomes

$$\begin{pmatrix} \cosh(\beta'z) & -e^\eta \sinh(\beta'z) \\ -e^{-\eta} \sinh(\beta'z) & \cosh(\beta'z) \end{pmatrix}. \quad (9.53)$$

If  $\alpha = \beta$ , the lower-left element of the  $G$  matrix has to vanish, and it becomes

$$G = \begin{pmatrix} 0 & -2\alpha \\ 0 & 0 \end{pmatrix}, \quad (9.54)$$

and the  $M$  matrix takes the triangular form

$$\begin{pmatrix} 1 & -2\alpha z \\ 0 & 1 \end{pmatrix}. \quad (9.55)$$

The optical material can be made to provide rotations of the polarization axis. It is much more interesting to see this additional effect of squeeze.

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# Chapter 10

## Poincaré sphere

While the Jones vector formalism is a concrete physical example of the two-by-two representation of the Lorentz group, it cannot tell whether two orthogonal components are coherent with each other. In order to address this coherency issue, we need the two-by-two coherency matrix consisting of four Stokes parameters.

These four Stokes parameters define the parameters for the three-dimensional Poincaré sphere. The traditional Poincaré sphere needs three parameters like the Euler angles. Then what role does the fourth Stokes parameter play? We shall show in this chapter, the radius of the Poincaré sphere can change.

Since the Stokes parameters are constructed from the two-component Jones vectors, which are transformed like the  $SL(2, c)$  spinors, the two-by-two coherency matrix should transform like the two-by-two form of the space-time four-vector discussed extensively in Sec. 1.2 and again in Chapter 3.

Thus, the Lorentz group should leave the determinant of the coherency matrix invariant, but the degree of coherency changes the determinant. Thus, the coherency is an extra-Lorentzian variable. We shall study its implications in Einstein's energy-momentum relation where the particle mass is a Lorentz-invariant quantity. Thus the symmetry group has to be extended to the  $O(3, 2)$  group discussed in Sec. 7.3.

### 10.1 Coherency matrix

In order to address the question of coherency between the two orthogonal electric fields, we study the coherency matrix defined as (Born and Wolf 1980, Brosseau 1998)

$$C = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad (10.1)$$

with

$$\langle \psi_i^* \psi_j \rangle = \frac{1}{T} \int_0^T \psi_i^*(t + \tau) \psi_j(t) dt, \quad (10.2)$$

where  $T$  for a sufficiently long time interval, is much larger than  $\tau$ . Then, those four elements become (Han *et al.* 1997)

$$\begin{aligned} S_{11} &= \langle \psi_1^* \psi_1 \rangle = a^2, & S_{12} &= \langle \psi_1^* \psi_2 \rangle = ab e^{-(\sigma+i\phi)}, \\ S_{21} &= \langle \psi_2^* \psi_1 \rangle = ab e^{-(\sigma-i\phi)}, & S_{22} &= \langle \psi_2^* \psi_2 \rangle = b^2. \end{aligned} \quad (10.3)$$

The diagonal elements are the absolute values of  $\psi_1$  and  $\psi_2$  respectively. The off-diagonal elements could be smaller than the product of  $\psi_1$  and  $\psi_2$ , if the two transverse components are not completely coherent. The  $\sigma$  parameter specifies the degree of decoherence. The system is completely coherent if  $\sigma = 0$ . It is totally incoherent if  $\sigma = \infty$ .

If we start with the Jones vector of the form of Eq. (9.1), the coherency matrix becomes

$$C = \begin{pmatrix} a^2 & ab e^{-(\sigma+i\phi)} \\ ab e^{-(\sigma-i\phi)} & b^2 \end{pmatrix}. \quad (10.4)$$

We are interested in the symmetry properties of this matrix. Since the transformation matrix applicable to the Jones vector is the two-by-two representation of the Lorentz group, we are particularly interested in the transformation matrices applicable to this coherency matrix.

The trace and determinant of the above coherency matrix are

$$\begin{aligned} \det(C) &= (ab)^2 (1 - e^{-2\sigma}), \\ \text{tr}(C) &= a^2 + b^2. \end{aligned} \quad (10.5)$$

Since  $e^{-\sigma}$  is always smaller than one, we can introduce an angle  $\chi$  defined as

$$\cos \chi = e^{-\sigma}, \quad (10.6)$$

and call it the “decoherence angle.” If  $\chi = 0$ , the decoherence is minimum, and it is maximum when  $\chi = 90^\circ$ . We can then write the coherency matrix of Eq. (10.4) as

$$C = \begin{pmatrix} a^2 & ab(\cos \chi)e^{-i\phi} \\ ab(\cos \chi)e^{i\phi} & b^2 \end{pmatrix}. \quad (10.7)$$

The degree of polarization is defined as (Saleh and Teich 2007)

$$f = \sqrt{1 - \frac{4 \det(C)}{(\text{tr}(C))^2}} = \sqrt{1 - \frac{4(ab)^2 \sin^2 \chi}{(a^2 + b^2)^2}}. \quad (10.8)$$

This degree is one if  $\chi = 0$ . When  $\chi = 90^\circ$ , it becomes

$$\frac{a^2 - b^2}{a^2 + b^2}. \quad (10.9)$$

Without loss of generality, we can assume that  $a$  is greater than  $b$ . If they are equal, this minimum degree of polarization is zero.

Under the influence of the Lorentz transformation defined for the Jones vectors of Chapter 9.1 and Sec. 1.2, this coherency matrix is transformed as

$$\begin{aligned} C' &= G C G^\dagger = \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \end{aligned} \quad (10.10)$$

Indeed, this is the Lorentz transformation defined in Sec. 1.2.

Then it is possible to write

$$\begin{aligned}
 S_0 &= \frac{S_{11} + S_{22}}{2}, & S_3 &= \frac{S_{11} - S_{22}}{2}, \\
 S_1 &= \frac{S_{12} + S_{21}}{2}, & S_2 &= \frac{S_{12} - S_{21}}{2},
 \end{aligned}
 \tag{10.11}$$

as a four-vector.

These four parameters are called Stokes parameters, and four-by-four transformations applicable to these parameters are widely known as Mueller matrices (Mueller 1943, Azam 1977, Brosseau 1998).

The Mueller matrices perform Lorentz transformations on the four Stokes parameters. The correspondence between the four-by-four and two-by-two representations was discussed in detail in Chapter 4.

Table 10.1: Polarization optics and special relativity sharing the same mathematics. Each matrix has its clear role in both optics and relativity. The determinant of the two-by-two matrix obtained from the Stokes vector or from the four-momentum remains invariant under Lorentz transformations. It is interesting to note that the decoherency parameter (least fundamental) in optics corresponds to the mass (most fundamental) in particle physics.

| Polarization Optics       | Transformation Matrix   | Particle Symmetry   |
|---------------------------|---|---------------------|
| Phase shift $\phi$        | $\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$                                 | Rotation around $z$ |
| Rotation around $z$       | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ | Rotation around $y$ |
| Squeeze along $x$ and $y$ | $\begin{pmatrix} e^{\mu/2} & 0 \\ 0 & e^{-\mu/2} \end{pmatrix}$                                     | Boost along $z$     |
| $\sin^2 \chi$             | Determinant   | $(mass)^2$          |

Since we can construct the Jones vector of Eq. (9.1) by making Lorentz transformations on the simpler form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} a \exp \{i(kz - \omega t)\} \\ a \exp \{i(kz - \omega t)\} \end{pmatrix},
 \tag{10.12}$$

we can now drop the amplitude  $a$  and work with the coherency matrix of the form

$$C = \begin{pmatrix} 1 & e^{-i\phi} \cos \chi \\ e^{i\phi} \cos \chi & 1 \end{pmatrix}.
 \tag{10.13}$$

The Stokes parameters are

$$\begin{aligned} S_0 &= 1, & S_3 &= 0, \\ S_1 &= (\cos \chi) \cos \phi, & S_2 &= (\cos \chi)(\sin \phi). \end{aligned} \quad (10.14)$$

The Poincaré sphere is defined by  $S_1$ ,  $S_2$ , and  $S_3$ .

$$R = \sqrt{S_1^2 + S_2^2 + S_3^2}. \quad (10.15)$$

Since  $S_3 = 0$  in Eq. (10.14), the sphere collapses into a circle of radius

$$R = \sqrt{S_1^2 + S_2^2} = \cos \chi. \quad (10.16)$$

Let us go back to the four-momentum matrix of Eq. (1.21) of Sec. 1.2. Its determinant is  $m^2$  and remains invariant under Lorentz transformations defined by Eq. (1.22). Likewise, the determinant of the coherency matrix of Eq. (10.4) should also remain invariant. The determinant in this case is

$$S_0^2 - R^2 = \sin^2 \chi. \quad (10.17)$$

However, this quantity depends on the angle  $\chi$  variable which measures decoherency of the two transverse components. This aspect is illustrated in Table 10.1.

While the decoherency parameter is not fundamental and is influenced by environment, it plays the same mathematical role as in the particle mass which remains as the most fundamental quantity since Isaac Newton, and even after Einstein.

## 10.2 Entropy problem

It is remarkable that the coherency matrix can also serve as the density matrix if it is divided by 2. It is written as

$$\rho(\chi) = \frac{C}{2} = \frac{1}{2} \begin{pmatrix} 1 & \cos \chi \\ \cos \chi & 1 \end{pmatrix}, \quad (10.18)$$

The trace of this density matrix is one.

This density matrix can be diagonalized to

$$\rho(\chi) = \frac{1}{2} \begin{pmatrix} 1 + \cos \chi & 0 \\ 0 & 1 - \cos \chi \end{pmatrix}. \quad (10.19)$$

Then the entropy becomes (Kim and Wigner 1990, Eisert *et al.* 2010, Kim and Noz 2014)

$$S(\chi) = -\frac{1 + \cos \chi}{2} \ln \left( \frac{1 + \cos \chi}{2} \right) - \frac{1 - \cos \chi}{2} \ln \left( \frac{1 - \cos \chi}{2} \right), \quad (10.20)$$

which can be simplified to

$$S(\chi) = -[\cos^2(\chi/2)] \ln[\cos^2(\chi/2)] - [\sin^2(\chi/2)] \ln[\sin^2(\chi/2)]. \quad (10.21)$$

If the entropy is zero the system is completely coherent with  $\sigma = \chi$ . The entropy takes the maximum value of  $\ln(2)$  when the system is totally incoherent with  $\chi = 90^\circ$  and  $\sigma = \infty$ .

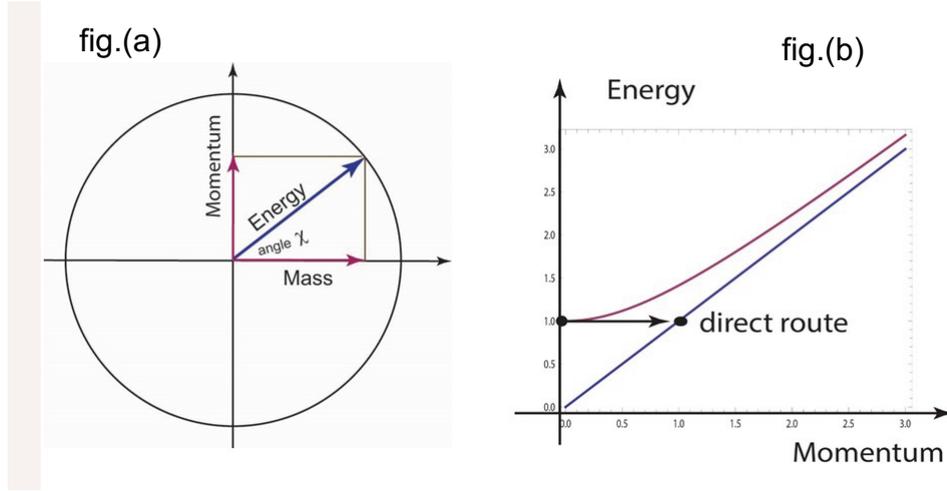


Figure 10.1: The momentum-mass relation for a fixed value of energy in fig.(a). As the angle  $\chi$  increases, the momentum increases, while the mass becomes smaller. The particle becomes massless for  $\chi = 90^\circ$ . This transition is described in terms of the traditional energy-momentum plot in fig.(b). This transition is not allowed within the framework of the Lorentz group. The Poincaré sphere allows this extra-Lorentz transformation.

### 10.3 Symmetries derivable from the Poincaré sphere

It has been demonstrated in Sec. 10.1 that the Poincaré sphere contains the symmetry of the Lorentz group applicable to the momentum-energy four-vector. While the Lorentz group cannot tolerate the variable mass, the sphere has an extra-Lorentz variable which can change the mass. In order to understand this extra variable, we exploit the symmetry between  $\cos \chi$  and  $\sin \chi$ , write another coherency matrix

$$C' = \begin{pmatrix} 1 & e^{-i\phi} \sin \chi \\ e^{i\phi} \sin \chi & 1 \end{pmatrix}, \quad (10.22)$$

with determinant  $\cos^2 \chi$ .

It is possible to diagonalize both  $C$  and  $C'$  to

$$C = \begin{pmatrix} 1 + \cos \chi & 0 \\ 0 & 1 - \cos \chi \end{pmatrix}, \quad C' = \begin{pmatrix} 1 + \sin \chi & 0 \\ 0 & 1 - \sin \chi \end{pmatrix}. \quad (10.23)$$

whose determinants are  $\sin^2 \chi$  and  $\cos^2 \chi$  respectively. The matrices  $C$  and  $C'$  share the same physics and the same mathematics. The choice is thus only for convenience.

We are now led to write the four-momentum matrix as

$$P = \begin{pmatrix} E + p & 0 \\ 0 & E - p \end{pmatrix}, \quad (10.24)$$

where  $E$  and  $p$  are the energy and the magnitude of the momentum respectively. The particle moves along the  $z$  direction.

Let us write  $p = E \cos \chi$ , and then the matrix  $P$  becomes

$$P = E \begin{pmatrix} 1 + \cos \chi & 0 \\ 0 & 1 - \cos \chi \end{pmatrix}. \quad (10.25)$$

This matrix is like  $C$  of Eq. (10.23). If we let  $p = E \sin \chi$ ,

$$P = E \begin{pmatrix} 1 + \sin \chi & 0 \\ 0 & 1 - \sin \chi \end{pmatrix} \quad (10.26)$$

is like  $C'$ .

Let us pick  $P$  of Eq. (10.26). If  $\chi = 0$ , the matrix becomes the four-momentum of the particle at rest with a mass  $E$ . If  $\chi = 90^\circ$ , the particle becomes massless with a momentum of  $E$ . The continuous transition from  $\chi = 0$  to  $90^\circ$  is illustrated in Fig. 10.2.

Starting from a massive particle at rest, we are interested in reaching a massless particle with the same energy. This problem is not new in that it was discussed in detail in Chapter 4 within the framework of the Lorentz group. However, the transition from the massive case to the massless case is a singular transformation, since the mass remains invariant under Lorentz transformations.

On the other hand, in the case of the Poincaré sphere, this transition is continuous as indicated in Fig. 10.1. The question is whether this extra-Lorentzian transformation can be accommodated by a larger symmetry group.

## 10.4 $O(3,2)$ symmetry

The group  $O(3, 2)$  is the Lorentz group applicable to a five-dimensional space applicable to three space dimensions and two time dimensions. Likewise, there are two energy variables, which lead to a five-component vector

$$(E_1, E_2, p_z, p_x, p_y). \quad (10.27)$$

In order to study this group, we have to use five-by-five matrices, but we are interested in its subgroups. First of all, there is a three-dimensional Euclidean space consisting of  $p_z, p_x$ , and  $p_y$ , to which the  $O(3)$  rotation group is applicable, as in the case of the  $O(3, 1)$  Lorentz group.

If the momentum is in the  $z$  direction, this five-vector becomes

$$(E_1, E_2, p, 0, 0). \quad (10.28)$$

As for these two energy variables, they take the form

$$E_1 = \sqrt{p^2 + m_1^2}, \quad \text{and} \quad E_2 = \sqrt{p^2 + m_2^2 \cos^2 \chi}, \quad (10.29)$$

with (Zee 2003, Başkal and Kim 2006)

$$m_1 = m \cos \chi, \quad m_2 = m \sin \chi, \quad (10.30)$$

and they maintain

$$E_1^2 + E_2^2 = m^2 + 2p^2, \quad (10.31)$$

which remains constant for a fixed value of  $p^2$ . There is thus a rotational symmetry in the two-dimensional space of  $E_1$  and  $E_2$ . Indeed, this defines the  $O(3, 2)$  de Sitter symmetry (Başkal and Kim 2006, 2013).

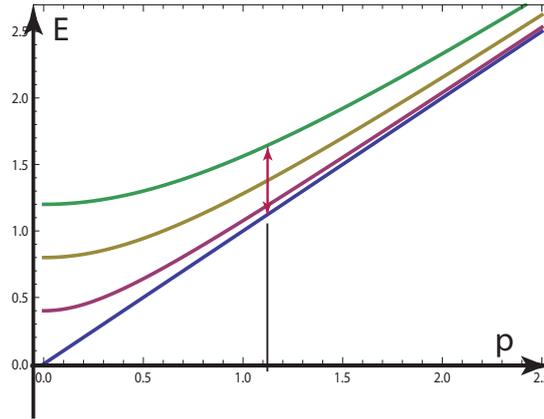


Figure 10.2: Energy-momentum hyperbolas for different values of the mass. The Lorentz group does not allow us to jump from one hyperbola to another, but it is possible within the framework of the  $O(3,2)$  de Sitter symmetry. This figure illustrates the transition while the magnitude of the momentum is kept constant.

For the present purpose, the most important subgroups are two Lorentz subgroups applicable to the Minkowskian spaces of

$$(E_1, p, 0, 0), \quad \text{and} \quad (E_2, p, 0, 0). \quad (10.32)$$

Then, in the two-by-two matrix representation, these four-momenta take the form

$$\begin{pmatrix} E_1 + p & 0 \\ 0 & E_1 - p \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} E_2 + p & 0 \\ 0 & E_2 - p \end{pmatrix}, \quad (10.33)$$

with determinants equal to  $m^2 \cos^2 \chi$  and  $m^2 \sin^2 \chi$  respectively.

With this understanding, we can now concentrate only on the matrix with  $E_1$ . For  $\chi = 0$ ,  $E_1 = p$ , and it takes a maximum value of  $\sqrt{p^2 + m^2}$ . This fixed-momentum variation is illustrated in Fig. 10.2. Indeed, the de Sitter symmetry allows us to jump from one mass hyperbola to another.

$O(3,2)$  or  $SO(3,2)$  appears as the symmetry group in various branches of physics, ranging from quantum mechanics to extended theories of gravity (Wesson 2006, Zee 2013). In Sec. 7.3, we have extensively discussed the quantum mechanical case in the context of the Dirac's harmonic oscillator (Dirac 1963). In addition to Dirac's oscillator, spin-orbit coupled harmonic oscillators also admit the same symmetry group (Haaker 2014).

In fact, emerging from different viewpoints, attempts to accommodate another time-like dimension to the usual spacetime is not new (Bars 1998, Wesson 2002).  $SO(3,2)$  is also the isometry group for the  $\text{AdS}^4$  (anti-de Sitter) spacetimes. It admits closed time-like curves and solves Einstein's equations with a negative cosmological constant accounting for a contracting universe.

On the other hand, the  $\text{dS}^4$  (de Sitter) spacetime whose isometry group is  $O(4,1)$  is in contrast with  $\text{AdS}^4$  in the sense that it solves Einstein's equations with a positive cosmological constant and accounts for the expansion of the universe which is what we are observing now in the real cosmos (Zee 2013). It is gratifying to note that the Poincaré sphere contains this important symmetry.

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