Time separation as a hidden variable to the Copenhagen school of quantum mechanics

Y. S. Kim* and M. E. Noz†

*1
Center for Fundamental Physics, University of Maryland, College Park, Maryland 20742, U.S.A.
†2
Department of Radiology, New York University, New York, New York 10016, U.S.A.

Abstract. The Bohr radius is a space-like separation between the proton and electron in the hydrogen atom. According to the Copenhagen school of quantum mechanics, the proton is sitting in the absolute Lorentz frame. If this hydrogen atom is observed from a different Lorentz frame, there is a time-like separation linearly mixed with the Bohr radius. Indeed, the time-separation is one of the essential variables in high-energy hadronic physics where the hadron is a bound state of the quarks, while thoroughly hidden in the present form of quantum mechanics. It will be concluded that this variable is hidden in Feynman's rest of the universe. It is noted first that Feynman's Lorentz-invariant differential equation for the bound-state quarks has a set of solutions which describe all essential features of hadronic physics. These solutions explicitly depend on the time separation between the quarks. This set also forms the mathematical basis for two-mode squeezed states in quantum optics, where both photons are observable, but one of them can be treated a variable hidden in the rest of the universe. The physics of this two-mode state can then be translated into the time-separation variable in the quark model. As in the case of the un-observed photon, the hidden time-separation variable manifests itself as an increase in entropy and uncertainty.

Keywords: hidden variables, Lorentz-covariant bound states, squeezed states of light
PACS: 03.65.Ge, 03.65.Pm

INTRODUCTION

The Bohr radius is a very important parameter in the present form of quantum mechanics. Niels Bohr spent much of his research life on the hydrogen atom. He also had a great respect for Einstein. Whenever he mentions "space" he also adds "time." Yet, he never thought about the proton in other than the absolute frame. Nor did Einstein raise this issue. At their time, it was beyond their imagination that bound-state particles could move with relativistic speed.

There are now composite particles moving with speed close to that of velocity of light, but they are not hydrogen atoms. They are protons coming out from particle accelerators, and each proton is a bound state of quarks. As far as quantum bound states are concerned, the hydrogen atom went through an evolutionary process as described in Fig. 1.

[1] electronic address: yskim@physics.umd.edu
[2] electronic address: nozm01@nyumc.org

Advances in Quantum Theory
AIP Conf. Proc. 1327, 138-147 (2011); doi: 10.1063/1.3567437

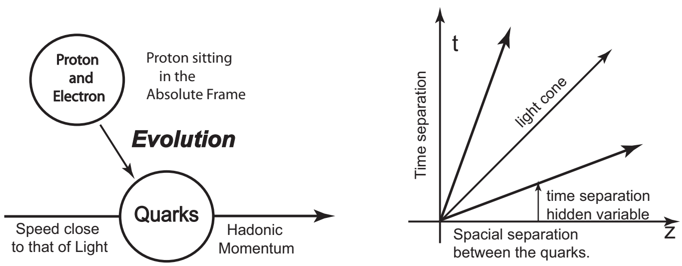

FIGURE 1. Evolution of the hydrogen atom. The Bohr radius measures the spacial separation between the proton and electron in the hydrogen atom, without time separation. Let us assume that this separation is zero in the frame where the hydrogen atom is at rest. Then the time-separation becomes prominent to a moving observer.

In this paper, we start with the Lorentz-invariant oscillator equation of Feynman *et al.* [1], and note that there is a Lorentz-covariant set of solutions representing Wigner's little group dictating the internal space-time symmetry of particles [2]. This set also describes the essential feature of high-energy hadronic physics [3]. Furthermore, these covariant solutions carry quantum probability interpretation [3].

It is noted also that this set constitutes the mathematical basis for two-photon coherent states or two-mode squeezed state, where both photons are observable [4]. This two-photon system allows us to consider when one of the photons is not observed [5].

The Lorentz-covariant oscillator formalism thus allows us to study what happens when the time-separation variable is not observed. We thus conclude that it is hidden in Feynman's rest of the universe [6, 7], which is well defined in terms of the two-mode squeezed state. Since this variable is hidden, it causes an increase in entropy and uncertainty.

In Sec. , we write down the Lorentz-invariant differential equation which Feynman *et al.* used to study the hadronic mass spectra [1]. In Sec. , it is shown that the Feynman equation has a set of solutions that can represent Wigner's $O(3)$-like little group for massive particles. In Sec. , it is shown that this set of oscillator solutions can combine Dirac's efforts to combine quantum mechanics and relativity [8, 9, 10]. It is shown that the same set of solutions can be obtained from the system of two coupled harmonic oscillators which forms the mathematical basis for two-mode squeezed states in quantum optics [4]. Finally, in Sec. , we illustrate Feynman's rest of universe using the coupled harmonic oscillators and two-mode squeezed states [6, 7]. We then show that the time-separation variable can be interpreted as one of the oscillator variables not observed. The result is an increase in statistical entropy and uncertainty, as is expected from hidden variables.

FEYNMAN'S LORENTZ-INVARIANT EQUATION

For solving practical problems in quantum mechanics, we use the Schrödinger wave equation. For scattering problems, we use running-wave solutions. For bound states, we obtain standing-wave solutions with their boundary conditions. Indeed, the localization boundary condition leads to discrete energy levels.

For scattering problems, we now have Lorentz-covariant quantum field theory with its scattering matrix formalism and Feynman diagrams. For bound state problems, there had been attempts in the past to understand bound-state problems using the S-matrix method. However, it was noted that the bound-state poles of the S-matrix do not always gurantee the localization of wave functions [11, 12].

In 1971, Feynman *et al.* [1] published a paper containing the Lorentz-invariant differential equation

$$\left\{-\frac{1}{2}\left[\left(\frac{\partial}{\partial x_\mu^a}\right)^2+\left(\frac{\partial}{\partial x_\mu^b}\right)^2\right]+\frac{1}{16}\left(x_\mu^a-x_\mu^b\right)^2+m_0^2\right\}\phi\left(x_\mu^a,x_\mu^b\right)=0, \tag{1}$$

for a hadron consisting of two quarks bound-together a harmonic oscillator potential. The space-time quark coordinates are x_μ^a and x_μ^b. They then wrote down the hadronic and the quark separation coordinates as

$$X_\mu=\frac{1}{2}\left(x_\mu^a+x_\mu^b\right), \qquad x_\mu=\frac{1}{2\sqrt{2}}\left(x_\mu^a-x_\mu^b\right), \tag{2}$$

respectively. We can now consider the solution of the form

$$\phi\left(x_\mu^a,x_\mu^b\right)=f\left(X_\mu\right)\psi\left(x_\mu\right), \tag{3}$$

where $f\left(X_m u\right)$ and $\psi\left(x_\mu\right)$ are for a free hadron and for the quarks inside the hadron respectively. $f\left(X_\mu\right)$ satisfies the Klein-Gordeon equation, and takes the form

$$f(X)=\exp\left(\pm iP\cdot X\right), \tag{4}$$

with

$$-P^2=m_0^2+(\lambda+1). \tag{5}$$

The quark wave function satisfies the differential equation

$$\frac{1}{2}\left\{-\left(\frac{\partial}{\partial x_\mu}\right)^2+x_\mu^2\right\}\psi\left(x_\mu\right)=(\lambda+1)\psi\left(x_\mu\right). \tag{6}$$

This differential equation of Eq.(6) is a Lorentz-invariant equation, but its solution can take different forms depending on the separable coordinate systems with their boundary conditions. The problem is to choose the set of solutions which can tell us physics properly based on the existing rules of quantum mechanics and relativity.

If we ignore the time-like variable Eq.(6), it becomes the Schrödinger-type equation for a harmonic oscillator. The problem is the existence of the time-like variable. If we ignore it, it is the equation for non-relativistic quantum mechanics, but it is not Lorentz-invariant. If we include it, we have to give a physical interpretation to this variable.

SOLUTIONS REPRESENTING WIGNER'S LITTLE GROUP

In 1979, Kim, Noz, and Oh published a paper on representations of the Poincaré group using a set of solutions of the oscillator equation of Eq.(6) [13]. Later in 1986, Kim and Noz in their book [3] noted that this set corresponds to a representation of Wigner's $O(3)$-like little group for massive particles. If a particle has a non-zero mass, there is a Lorentz frame in which the particle is at rest. Wigner's little group then becomes that of group $O(3)$ which is the three-dimensional rotation group [2].

The solution of the Lorentz-invariant equation contains both space-like and time-like wave functions, but we can keep the time-like component to its ground state, in accordance with Dirac's c-number time-energy uncertainty relation [8]. The wave function still retains the $O(3)$-like symmetry. The solution takes the form

$$\psi(x,y,z,t) = \left[\left(\frac{1}{\pi}\right)^{1/4} \exp\left(\frac{-t^2}{2}\right)\right] \psi(x,y,x). \tag{7}$$

As for the spatial part of the differential equation, we note that it is the equation for the three-dimensional oscillator. We can solve this equation with both the Cartesian and spherical coordinates. If we use the spherical system with (r,θ,ϕ) as the variables, the solution should take the form

$$\psi(x,y,z) = R_{\lambda,\ell}(r) Y_{\ell,m}(\theta,\phi) \exp\left\{-\left(\frac{x^2+y^2+z^2}{2}\right)\right\}, \tag{8}$$

where $Y_{\ell,m}(\theta,\phi)$ is the spherical harmonics, and $R_{\lambda,\ell}(r)$ is the normalized radial wave function with $r = \sqrt{x^2+y^2+z^2}$. The λ and ℓ parameters specify the mass and the internal spin of the hadron respectively, as required by Wigner's representation theory [2, 3].

This oscillator wave function is separable also in the Cartesian coordinates, and the solution can be written

$$\psi(x,y,z) = \left[\frac{1}{\pi\sqrt{\pi}2^{(a+b+n)}a!b!n!}\right]^{1/2} H_a(x)H_b(y)H_n(z) \exp\left\{-\left(\frac{x^2+y^2+z^2}{2}\right)\right\}, \tag{9}$$

where $H_n(z)$ is the Hermite polynomial, and λ of Eq.(6) is $(a+b+n)$.

When we boost this solution along the z direction, the Cartesian form of Eq.(9) is more convenient. Since the transverse x and y coordinates are not affected by this boost, we can separate out these variables in the oscillator differential equation of Eq.(6), and consider the differential equation

$$\frac{1}{2}\left\{\left[-\left(\frac{\partial}{\partial z}\right)^2 + z^2\right] - \left[-\left(\frac{\partial}{\partial t}\right)^2 + t^2\right]\right\} \psi(z,t) = n\psi(z,t). \tag{10}$$

This differential equation remains invariant under the Lorentz boost

$$z \to (\cosh\eta)z + (\sinh\eta)t, \qquad t \to (\sinh\eta)z + (\cosh\eta)t. \tag{11}$$

In terms of the hadronic velocity v, e^{η} takes the form

$$e^{\eta} = \sqrt{\frac{1+v/c}{1-v/c}}. \tag{12}$$

If we suppress the excitations along the t coordinate, the normalized solution of this differential equation is

$$\psi(z,t) = \left(\frac{1}{\pi 2^n n!}\right)^{1/2} H_n(z) \exp\left\{-\left(\frac{z^2+t^2}{2}\right)\right\}. \tag{13}$$

If we boost the hadron along the z direction, the coordinate variables z and t should be replaced respectively by z' and t' of Eq.(11), and the wave function becomes uncontrollable.

DIRAC'S ATTEMPTS TO COMBINE QUANTUM MECHANICS AND SPECIAL RELATIVITY

Paul A. M. Dirac published a number of important papers on combining quantum mechanics with relativity. In 1927 [8], Dirac noted that there is a time-energy uncertainty relation without time-like excitations. He pointed out that this space-time asymmetry causes a difficulty in combining quantum mechanics with special relativity.

In 1945 [9], Dirac constructed four-dimensional harmonic oscillator wave functions including the time variable. His oscillator wave functions took normalizable Gaussian form, but he did not attempt to give a physical interpretation to this mathematical device.

It is remarkable that the oscillator representation given in Sec. addresses Dirac's concerns in all of his papers mentioned above. In his 1949 paper [10], Dirac introduced his light-cone variables defined as

$$u = \frac{z+t}{\sqrt{2}}, \qquad v = \frac{z-t}{\sqrt{2}}. \tag{14}$$

Then the boost transformation of Eq.(11) takes the form

$$u \to e^{\eta} u, \qquad v \to e^{-\eta} v. \tag{15}$$

The u variable becomes expanded while the v variable becomes contracted. Their product

$$uv = \frac{1}{2}(z+t)(z-t) = \frac{1}{2}\left(z^2 - t^2\right) \tag{16}$$

remains invariant. Indeed, in Dirac's picture, the Lorentz boost is a squeeze transformation.

In this new notation, the wave function of Eq.(13) takes the form

$$\psi_{\eta}^{n}(x,t) = \left[\frac{1}{\pi n! 2^n}\right]^{1/2} H_n\left(\frac{e^{-\eta}u + e^{\eta}v}{\sqrt{2}}\right) \exp\left\{-\left(\frac{e^{-2\eta}u^2 + e^{2\eta}v^2}{2}\right)\right\}, \tag{17}$$

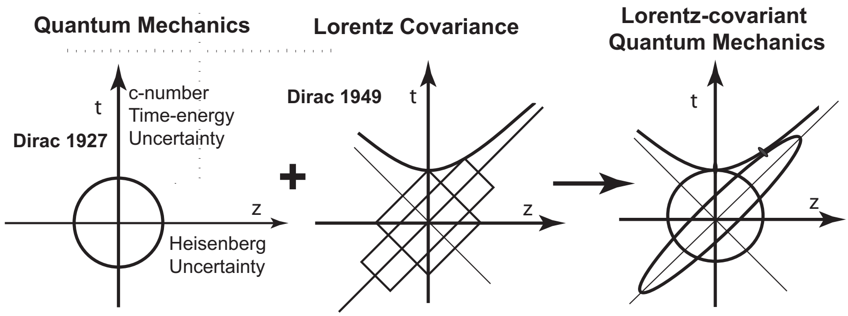

FIGURE 2. Lorentz-squeezed hadrons. Feynman's proposal leads us to combine Dirac's quantum mechanics and his light-cone representation of Lorentz boost to generate Lorentz-squeezed hadrons.

for the moving hadron. The Gaussian factor in this expression determines the space-time localization property of all excited-state wave functions. We can now combine Dirac's 1927 [8], 1945 [9], and 1949 [10] papers into Fig. 2.

This squeeze property has been experimentally verified in various observations in high-energy physics, including Feynman's parton picture [14, 15, 16].

In 1963 [17], Dirac used two coupled oscillators to understand Lorentz transformations. Following the spirit of Dirac, let us start with two independent oscillators. The wave function for this system is

$$\psi(x_1, x_2) = \chi_{n_1}(x_1)\,\chi_{n_2}(x_2), \tag{18}$$

where $\chi_n(z)$ is the n-th excited-state oscillator wave function which takes the form

$$\chi_n(x) = \left[\frac{1}{\sqrt{\pi}2^n n!}\right]^{1/2} H_n(x) \exp\left(\frac{-x^2}{2}\right). \tag{19}$$

If the x_2 coordinate is in its ground state, the wave function becomes

$$\psi(x_1, x_2) = \chi_n(x_1)\,\chi_0(x_2) = \left[\frac{1}{\pi 2^n n!}\right]^{1/2} H_n(x_1) \exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right], \tag{20}$$

with $n = n_1$. In order to couple these two oscillators, we introduce the normal coordinates

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \qquad y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2), \tag{21}$$

and squeeze these variables:

$$y_1 \to e^{\eta} y_1, \qquad y_2 \to e^{-\eta} y_2, \tag{22}$$

Then the squeezed wave function takes the form of Eq.(17) with u and v replaced by y_1 and y_2 respectively, or with z and t by x_1 and x_2 respectively. Furthermore, this wave function can be expanded in terms of $\chi_n(x)$ [4]. If $n = 0$ in Eq.(20), the wave function becomes Gaussian, and its squeezed form becomes

$$\psi_{\eta}^{0}(x_1, x_2) = \left(\frac{1}{\cosh\eta}\right)\sum_k (\tanh\eta)^k \chi_k(x_1)\,\chi_k(x_2)\,. \tag{23}$$

This expression is for the two-mode squeezed states in quantum optics [4, 17], where $\chi(x_1)$ and $\chi(x_2)$ are the states of the first and second photons respectively. Using this formula, it is possible to study what happens if the second photon is not observed [5].

HIDDEN IN FEYNMAN'S REST OF THE UNIVERSE

Throughout this paper, the time-separation variable played a major role in the covariant formulation of the harmonic oscillator wave functions. It should exist wherever the space separation exists. The Bohr radius is the measure of the separation between the proton and electron in the hydrogen atom. If this atom moves, the radius picks up the time separation, according to Einstein [18].

On the other hand, the present form of quantum mechanics does not include this time-separation variable. The best way we can do at the present time is to treat this time-separation as a variable in Feynman's rest of the universe [7]. In his book on statistical mechanics [6], Feynman states

> *When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

This abstract statement can be studied in terms of two coupled oscillators [7], and also in terms of two-mode squeezed states [5]. The failure to include what happens outside the system results in an increase of entropy. The entropy is a measure of our ignorance and is computed from the density matrix. The density matrix is needed when the experimental procedure does not analyze all relevant variables to the maximum extent consistent with quantum mechanics. If we do not take into account the time-separation variable, the result is therefore an increase in entropy [19, 20].

From the covariant oscillator wave functions defined in this section, the pure-state density matrix is

$$\rho_{\eta}^{n}(z,t;z',t') = \psi_{\eta}^{n}(z,t)\,\psi_{\eta}^{n}(z',t'), \tag{24}$$

which satisfies the condition $\rho^2 = \rho$:

$$\rho_{\eta}^{n}(z,t;x',t') = \int \rho_{\eta}^{n}(z,t;x",t")\rho_{\eta}^{n}(z",t";z',t')dz"dt". \tag{25}$$

In order to simplify our discussion without sacrificing physics, we carry out our calculation for the ground state only with $n = 0$. The computation can be extended for excited states.

Since we are not measuring the time-separation variable, we have to take the trace of the matrix with respect to the t variable. The resulting density matrix is

$$\rho(z,z') = \left(\frac{1}{\pi\cosh(2\eta)}\right)^{1/2} \exp\left\{-\frac{1}{4}\left[\frac{(z+z')^2}{\cosh(2\eta)} + (z-z')^2\cosh(2\eta)\right]\right\}, \tag{26}$$

The standard way to measure this ignorance is to calculate the entropy defined as

$$S = -Tr\left(\rho\ln(\rho)\right), \tag{27}$$

which, for density matrix of Eq.(26), becomes

$$S = (\cosh^2\eta)\ln(\cosh^2\eta) - (\sinh^2\eta)\ln(\sinh^2\eta). \tag{28}$$

The quark distribution $\rho(z,z)$ becomes

$$\rho(z,z) = \left(\frac{1}{\pi\cosh(2\eta)}\right)^{1/2} \exp\left(\frac{-z^2}{\cosh(2\eta)}\right). \tag{29}$$

The width of the distribution becomes $\sqrt{\cosh\eta}$, and becomes wide-spread as the hadronic speed increases. Likewise, the momentum distribution becomes wide-spread [3]. This effect can be seen from the Wigner phase-space distribution function defined as

$$W(z,p) = \int \rho(z+y,z-y)e^{2ipy}dy. \tag{30}$$

For the density matrix of Eq.(29), this Wigner function becomes

$$W(z,p) = \frac{1}{\cosh(2\eta)} \exp\left\{-\left(\frac{z^2+p^2}{\cosh(2\eta)}\right)\right\}. \tag{31}$$

This position-momentum distribution is illustrated in Fig.(3).

If the hadron is at rest, the time-separation variable does not play any role in the system. The uncertainty is purely from Heisenberg's uncertainty relation. If the hadron moves, and if we do not observe the time-separation variable, there is an added uncertainty as described in Fig. 3. This is exactly what we expect from hidden variables. Indeed, the time-separation variable is hidden in Feynman's rest of the universe.

Let us go back to Eq.(23) of Sec. . This is a series expansion of the squeezed ground state wave function. This formula serves as the two-photon coherent state in quantum optics. If we do not observe one of the photons, the mathematics is exactly the same as the one we carried out for the Lorentz-squeezed hadron we presented in this report. The increase in entropy and uncertainty in this case has been discussed in the literature [5]. This example allows us to study the hidden time-separation variable in terms of what we observe in the real world.

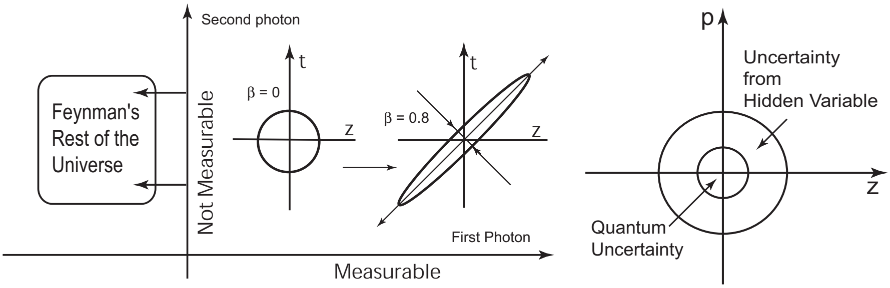

FIGURE 3. Probability distribution of the two-oscillator system, which can also be used for the covariant harmonic oscillators and the two-photon coherent states. One of the coordinates is observed and the other is is hidden in Feynman's rest of the universe. In the phase-space picture of quantum mechanics, the small circle indicates the minimal uncertainty when the hadron is at rest. The statistical uncertainty is added when the hadron moves. This is illustrated by a larger circle. The radius of this circle increases by $\sqrt{\cosh(2\eta)}$ as the hadron picks up the speed while the time-separation varible remains as a hidden variable.

CONCLUDING REMARKS

In Einstein's Lorentz-covariant world, the time separation exists whenever there is a space separation like the Bohr radius. However, this variable is never mentioned in the Copenhagen interpretation of quantum mechanics.

In order to see what happens if this variable is included, we started with Feynman's phenomenological equation for the quarks inside the moving hadron. It is shown that there is a set of solutions possessing the symmetry of Wigner's little group dictating internal space-time symmetry of particles in the Lorentz-covariant world.

It is noted that this set of solutions constitutes the mathematical basis for two-photon coherent states, where both photons are observable, but we can also study the case where one of them is not observable.

With these tools, we have shown that the time-separation variable is hidden in Feynman's rest of the universe. This causes an increase in entropy and uncertainty, as we expect from hidden variables.

At this time, we are not able to say anything about possible hidden variables behind Heisenberg's uncertainty principle illustrated by a small circle in the phase-space picture in Fig 3. This could be a similar problem or an entirely different problem. We do not know.

REFERENCES

1. R. P. Feynman, M. Kislinger, and F. Ravndal, *Phys. Rev. D* **3**, 2706–2732 (1971).
2. E. Wigner, *Ann. Math.* **40**, 149–204 (1939).
3. Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group*, Reidel, Dordrecht, 1986.
4. Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* World Scientific, Singapore, 1991.
5. A. K. Ekert and P. L. Knight, *Am. J. Phys.* **57**, 692–697 (1989).

6. R. P. Feynman, *Statistical Mechanics* Benjamin/Cummings, Reading, Massachusetts, 1972.
7. D. Han, Y. S. Kim, and M. E. Noz, *Am. J. Phys.* **67**, 61–66 (1999).
8. P. A. M. Dirac, *Proc. Roy. Soc. (London)* **A114**, 243–265 (1927).
9. P. A. M. Dirac, *Proc. Roy. Soc. (London)* **A183**, 284–295 (1945).
10. P. A. M. Dirac, *Rev. Mod. Phys.* **21**, 392–399 (1949).
11. Y. S. Kim, *Phys. Rev.* **142**, 1150–1153 (1966).
12. G. F. Chew, *Phys. Rev. Lett.* **19**, 1492–1495 (1967).
13. Y. S. Kim, M. E. Noz, and S. H. Oh, *J. Math. Phys.* **20**, 1341–1344 (1979).
14. R. P. Feynman, *Phys. Rev. Lett.* **23**, 1415–1417 (1969).
15. Y. S. Kim and M. E. Noz, *Phys. Rev. D* **15**, 335–338 (1977).
16. Y. S. Kim, *Phys. Rev. Lett.* **63**, 348–351 (1989).
17. P. A. M. Dirac, *J. Math. Phys.* **4**, 901–909 (1963).
18. Y. S. Kim and M. E. Noz, The Question of Simultaneity in Relativity and Quantum Mechanics. in *QUANTUM THEORY: Reconsideration of Foundations - 3* edited by G. Adenier, A. Khrennikov, and T. M. Nieuwenhuizen, AIP Conference Proceedings 180, American Institute of Physics, College Park, MD, 2006, pp. 168–178.
19. Y. S. Kim and E. P. Wigner, *Phys. Lett. A* **147**, 343–347 (1990).
20. Y. S. Kim, *J. Phys. Conf. Ser.* **70**, 012010 (2007).