Harmonic Oscillators as Bridges between Theories

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Abstract. Other than scattering problems where perturbation theory is applicable, there are basically two ways to solve problems in physics. One is to reduce the problem to harmonic oscillators, and the other is to formulate the problem in terms of two-by-two matrices. If two oscillators are coupled, the problem combines both two-by-two matrices and harmonic oscillators. This method then becomes a powerful research tool to cover many different branches of physics. Indeed, the concept and methodology in one branch of physics can be translated into another through the common mathematical formalism. It is noted that the present form of quantum mechanics is largely a physics of harmonic oscillators. Special relativity is the physics of the Lorentz group which can be represented by the group of by two-by-two matrices commonly called $SL(2,c)$. Thus the coupled harmonic oscillators can therefore play the role of combining quantum mechanics with special relativity. Both Paul A. M. Dirac and Richard P. Feynman were fond of harmonic oscillators, while they used different approaches to physical problems. Both were also keenly interested in making quantum mechanics compatible with special relativity. It is shown that the coupled harmonic oscillators can bridge these two different approaches to physics.

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INTRODUCTION

Because of its mathematical simplicity, the harmonic oscillator provides soluble models in many branches of physics. It often gives a clear illustration of abstract ideas. In many cases, the problems are reduced to the problem of two coupled oscillators. Soluble models in quantum field theory, such as the Lee model [1] and the Bogoliubov transformation in superconductivity [2], are based on two coupled oscillators. More recently, the coupled oscillators form the mathematical basis for squeezed states in quantum optics [3].

According to our experience, the present form of quantum mechanics is largely a physics of harmonic oscillators. Since the group $SL(2,C)$ forms the universal covering group of the Lorentz group, special relativity is a physics of two-by-two matrices. Therefore, the coupled harmonic oscillator can provide a concrete model for relativistic quantum mechanics.

With this point in mind, Dirac and Feynman used harmonic oscillators to test their physical ideas. In this paper, we first examine Dirac's attempts to combine quantum mechanics with relativity in his own style: to construct mathematically appealing models. We then examine how Feynman approached this problem. He was insisting on his own style. Observe the experimental world, tell the story of the real world, and then write down mathematical formulas as needed.

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In this paper, we use coupled harmonic oscillators to build a bridge between the two different attempts made by Dirac and Feynman.

In section 1, we start with the classical Hamiltonian for two coupled oscillators. It is possible to obtain a explicit solution for the Schrödinger equation in terms of the normal coordinates.

Section 2 examines Dirac's life-long efforts to combine quantum mechanics and special relativity. Starting from Dirac's work, we construct a covariant model of relativistic extended particles by combining Dirac's oscillators with Feynman's phenomenological approach to relativistic quark model. In section 3, it is shown that Feynman's parton model can be interpreted as a limiting case of one Lorentz-covariant bound-state model.

COUPLED OSCILLATORS

Two coupled harmonic oscillators serve many different purposes in physics. It is well known that this oscillator problem can be formulated into a problem of a quadratic equation in two variables. The diagonalization of the quadratic form includes a rotation of the coordinate system. However, the diagonalization process requires additional transformations involving the scales of the coordinate variables [4].

In this paper, we start with a simple problem of two oscillators with equal mass. This contains enough physics for our present purpose. Then the Hamiltonian takes the form

$$H = \frac{1}{2}\left\{\frac{1}{m}p_1^2 + \frac{1}{m}p_2^2 + Ax_1^2 + Ax_2^2 + 2Cx_1x_2\right\}. \tag{1}$$

If we choose coordinate variables

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \qquad y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2), \tag{2}$$

the Hamiltonian can be written as

$$H = \frac{1}{2m}\left\{p_1^2 + p_2^2\right\} + \frac{K}{2}\left\{e^{-2\eta}y_1^2 + e^{2\eta}y_2^2\right\}, \tag{3}$$

where

$$K = \sqrt{A^2 - C^2}, \qquad \exp(2\eta) = \sqrt{\frac{A-C}{A+C}}, \tag{4}$$

If y_1 and y_2 are measured in units of $(mK)^{1/4}$, the ground-state wave function of this oscillator system is

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}}\exp\left\{-\frac{1}{2}(e^{-\eta}y_1^2 + e^{\eta}y_2^2)\right\}, \tag{5}$$

The wave function is separable in the y_1 and y_2 variables. However, for the variables x_1 and x_2, the story is quite different [4]. The key question is how the quantum mechanics in the world of the x_1 variable is affected by the x_2 variable. If the x_2 space is not observed, it corresponds to Feynman's rest of the universe [4].

Let us write the wave function of Eq.(5) in terms of x_1 and x_2, then

$$\psi_\eta(x_1,x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left[e^{-\eta}(x_1+x_2)^2 + e^{\eta}(x_1-x_2)^2\right]\right\}. \tag{6}$$

When the system is decoupled with $\eta = 0$, this wave function becomes

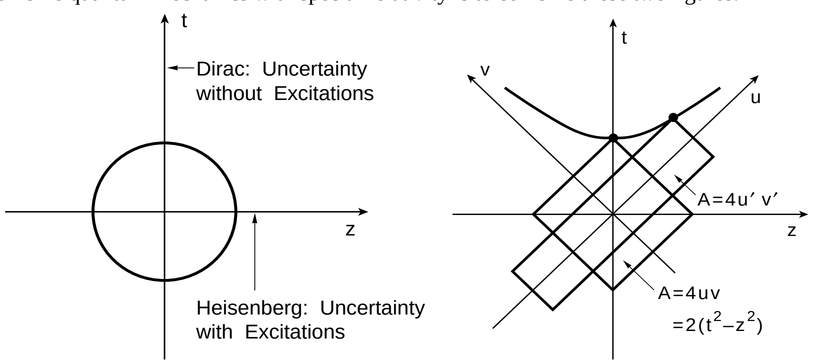
$$\psi_0(x_1,x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(x_1^2+x_2^2)\right\}. \tag{7}$$

The system becomes separable and becomes decoupled.

DIRAC'S HARMONIC OSCILLATORS

Paul A. M. Dirac is known to us through the Dirac equation for spin-1/2 particles. But his main interest was in the foundational problems. First, Dirac was never satisfied with the probabilistic formulation of quantum mechanics. This is still one of the hotly debated subjects in physics. Second, if we tentatively accept the present form of quantum mechanics, Dirac was insisting that it has to be consistent with special relativity. He wrote several important papers on this subject. Let us look at some of his papers on this subject.

TABLE 1. Quantum qechanics and special relativity. There are quantum excitations along the space-like longitudinal direction, but there are no excitations along the time-like direction. The time-energy relation is a c-number uncertainty relation. As for special relativity, Dirac's light-cone system leads to a squeeze transformation illustrated in this table. One way to combine quantum mechanics with special relativity is to combine these two figures.



During World War II, Dirac was looking into the possibility of constructing representations of the Lorentz group using harmonic oscillator wave functions [5]. The Lorentz group is the language of special relativity, and the present form of quantum mechanics starts with harmonic oscillators. Presumably, therefore, he was interested in making quantum mechanics Lorentz-covariant by constructing representations of the Lorentz group using harmonic oscillators.

In his 1945 paper [5], Dirac considers the Gaussian form

$$\exp\left\{-\frac{1}{2}\left(z^2+t^2\right)\right\}, \tag{8}$$

where z and t are the longitudinal and time-like variables respectively. This is a strange expression for those who believe in Lorentz invariance. The expression $\left(z^2-t^2\right)$ is invariant, but Dirac's Gaussian form of Eq.(8) is not. Yet, Dirac's expression of Eq.(8) is consistent with his earlier papers on the time-energy uncertainty relation [6]. In those papers, Dirac observed that there is a time-energy uncertainty relation, while there are no excitations along the time axis. He called this the "c-number time-energy uncertainty" relation. When one of us (YSK) was talking with Dirac in 1978, he clearly mentioned this word again. He said further that this is one of the stumbling block in combining quantum mechanics with relativity. This situation is illustrated in Table 1.

In 1949, the Reviews of Modern Physics published a special issue to celebrate Einstein's 70th birthday. This issue contains Dirac paper entitled "Forms of Relativistic Dynamics" [7]. In this paper, he introduced his light-cone coordinate system, in which a Lorentz boost becomes a squeeze transformation. When the system is boosted along the z direction, the transformation takes the form

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}. \tag{9}$$

This is not a rotation, and people still feel strange about this form of transformation. In 1949 [7], Dirac introduced his light-cone variables defined as [7]

$$u = (z+t)/\sqrt{2}, \qquad v = (z-t)/\sqrt{2}, \tag{10}$$

the boost transformation of Eq.(9) takes the form

$$u' = e^{\eta/2}u, \qquad v' = e^{-\eta/2}v. \tag{11}$$

The u variable becomes expanded while the v variable becomes contracted, as is illustrated in Table 1. Their product uv remains invariant. In Dirac's picture, the Lorentz boost is a squeeze transformation.

If we combine the two figures in Table 1, we end up with Fig. 1. In mathematical formulae, this transformation changes the Gaussian form of Eq.(8) into

$$\psi_\eta(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(e^{-\eta}u^2 + e^{\eta}v^2\right)\right\}. \tag{12}$$

Let us go back to section 1 on the coupled oscillators. The above expression is the same as Eq.(5). The x_1 variable now became the longitudinal variable z, and the x_2 variable became the time like variable t.

We can use the coupled harmonic oscillators as the starting point of relativistic quantum mechanics. This allows us to translate the quantum mechanics of two coupled oscillators defined over the space of x_1 and x_2 into the quantum mechanics defined over the space time region of z and t.

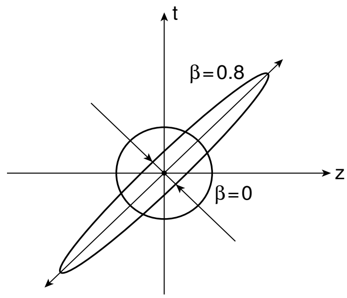


FIGURE 1. Effect of the Lorentz boost on the space-time wave function. The circular space-time distribution in the rest frame becomes Lorentz-squeezed to become an elliptic distribution.

This form becomes Eq.(8) when η becomes zero. The transition from Eq.(8) to Eq.(12) is a squeeze transformation. It is now possible to combine what Dirac observed into a covariant formulation of harmonic oscillator system. First, we can combine his c-number time-energy uncertainty relation described in Table 1 and his light-cone coordinate system given in the same table into a picture of covariant space-time localization illustrated in Fig. 1.

In addition, there are two more homework problems which Dirac left us to solve. First, in defining the t variable for the Gaussian form of Eq.(8), Dirac did not specify the physics of this variable. If it is going to be the calendar time, this form vanishes in the remote past and remote future. We are not dealing with this kind of object in physics. What is then the physics of this time-like t variable?

The Schrödinger quantum mechanics of the hydrogen atom deals with localized probability distribution. Indeed, the localization condition leads to the discrete energy spectrum. Here, the uncertainty relation is stated in terms of the spatial separation between the proton and the electron. If we believe in Lorentz covariance, there must also be a time-separation between the two constituent particles, and an uncertainty relation applicable to this separation variable. Dirac did not say in his papers of 1927 and 1945, but Dirac's "t" variable is applicable to this time-separation variable. This time-separation variable will be discussed in detail in section 3 for the case of relativistic extended particles.

Second, as for the time-energy uncertainty relation. Dirac'c concern was how the c-number time-energy uncertainty relation without excitations can be combined with uncertainties in the position space with excitations. Dira's 1927 paper was written before Wigner's 1939 paper on the internal space-time symmetries of relativistic particles [8].

Both of these questions can be answered in terms of the space-time symmetry of bound states in the Lorentz-covariant regime. In his 1939 paper, Wigner worked out internal space-time symmetries of relativistic particles [8]. He approached the problem by constructing the maximal subgroup of the Lorentz group whose transformations leave the given four-momentum invariant. As a consequence, the internal symmetry of a massive particle is like the three-dimensional rotation group.

If we extend this concept to relativistic bound states, the space-time asymmetry which Dirac observed in 1927 is quite consistent with Einstein's Lorentz covariance. The time variable can be treated separately. Furthermore, it is possible to construct a representations of Wigner's little group for massive particles [9]. As for the time-separation, it is also a variable governing internal space-time symmetry which can be linearly mixed when the system is Lorentz-boosted.

FEYNMAN'S OSCILLATORS

Quantum field theory has been quite successful in terms of Feynman diagrams based on the S-matrix formalism, but is useful only for physical processes where a set of free particles becomes another set of free particles after interaction. Quantum field theory does not address the question of localized probability distributions and their covariance under Lorentz transformations. In order to address this question, Feynman *et al.* suggested harmonic oscillators to tackle the problem [10]. Their idea is indicated in Fig. 2.

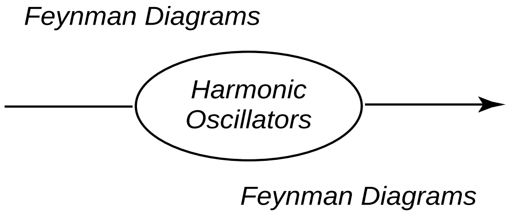


FIGURE 2. Feynman's roadmap for combining quantum mechanics with special relativity. Feynman diagrams work for running waves, and they provide a satisfactory resolution for scattering states in Einstein's world. For standing waves trapped inside an extended hadron, Feynman suggested harmonic oscillators as the first step.

Before 1964 [11], the hydrogen atom was used for illustrating bound states. These days, we use hadrons which are bound states of quarks. Let us use the simplest hadron consisting of two quarks bound together with an attractive force, and consider their space-time positions x_a and x_b, and use the variables

$$X = (x_a + x_b)/2, \qquad x = (x_a - x_b)/2\sqrt{2}. \tag{13}$$

The four-vector X specifies where the hadron is located in space and time, while the variable x measures the space-time separation between the quarks. According to Einstein, this space-time separation contains a time-like component which actively participates as in Eq.(9), if the hadron is boosted along the z direction. This boost can be conveniently described by the light-cone variables defined in Eq(10). Does this time-separation variable exist when the hadron is at rest? Yes, according to Einstein. In the present form of quantum mechanics, we pretend not to know anything about this variable. Indeed, this variable belongs to Feynman's rest of the universe.

What do Feynman *et al.* say about this oscillator wave function? In their classic 1971 paper [10], Feynman *et al.* start with the following Lorentz-invariant differential equation.

$$\frac{1}{2}\left\{x_{\mu}^{2}-\frac{\partial^{2}}{\partial x_{\mu}^{2}}\right\}\psi(x)=\lambda\,\psi(x). \tag{14}$$

This partial differential equation has many different solutions depending on the choice of separable variables and boundary conditions. Feynman *et al.* insist on Lorentz-invariant solutions which are not normalizable. On the other hand, if we insist on normalization, the ground-state wave function takes the form of Eq.(8). It is then possible to construct a representation of the Poincaré group from the solutions of the above differential equation [9]. If the system is boosted, the wave function becomes given in Eq.(12).

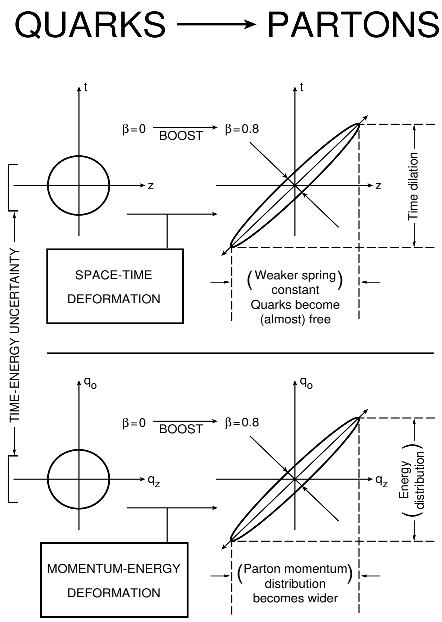


FIGURE 3. Lorentz-squeezed space-time and momentum-energy wave functions. As the hadron's speed approaches that of light, both wave functions become concentrated along their respective positive light-cone axes. These light-cone concentrations lead to Feynman's parton picture.

This wave function becomes Eq.(8) if η becomes zero. The transition from Eq.(8) to Eq.(12) is a squeeze transformation. The wave function of Eq.(8) is distributed within a circular region in the *uv* plane, and thus in the *zt* plane. On the other hand, the wave function of Eq.(12) is distributed in an elliptic region with the light-cone axes as the major and minor axes respectively. If η becomes very large, the wave function becomes concentrated along one of the light-cone axes. Indeed, the form given in Eq.(12) is a Lorentz-squeezed wave function. This squeeze mechanism is illustrated in Fig. 1.

There are many different solutions of the Lorentz invariant differential equation of Eq.(14). The solution given in Eq.(12) is not Lorentz invariant but is covariant. It is normalizable in the t variable, as well as in the space-separation variable z. It is indeed possible to construct Wigner's $O(3)$-like little group for massive particles [8], and thus the representation of the Poincaré group [9]. Our next question is whether this formalism has anything to do with the real world.

In 1969, Feynman observed that a fast-moving hadron can be regarded as a collection of many "partons" whose properties appear to be quite different from those of the quarks [12]. For example, the number of quarks inside a static proton is three, while the number of partons in a rapidly moving proton appears to be infinite. The question then is how the proton looking like a bound state of quarks to one observer can appear different to an observer in a different Lorentz frame? Feynman made the following systematic observations.

a. The picture is valid only for hadrons moving with velocity close to that of light.
b. The interaction time between the quarks becomes dilated, and partons behave as free independent particles.
c. The momentum distribution of partons becomes widespread as the hadron moves fast.
d. The number of partons seems to be infinite or much larger than that of quarks.

Because the hadron is believed to be a bound state of two or three quarks, each of the above phenomena appears as a paradox, particularly b) and c) together.

In order to resolve this paradox, let us write down the momentum-energy wave function corresponding to Eq.(12). If we let the quarks have the four-momenta p_a and p_b, it is possible to construct two independent four-momentum variables [10]

$$P = p_a + p_b, \qquad q = \sqrt{2}(p_a - p_b), \tag{15}$$

where P is the total four-momentum. It is thus the hadronic four-momentum.

The variable q measures the four-momentum separation between the quarks. Their light-cone variables are

$$q_u = (q_0 - q_z)/\sqrt{2}, \qquad q_v = (q_0 + q_z)/\sqrt{2}. \tag{16}$$

The resulting momentum-energy wave function is

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(e^{\eta} q_u^2 + e^{-\eta} q_v^2\right)\right\}. \tag{17}$$

Because we are using here the harmonic oscillator, the mathematical form of the above momentum-energy wave function is identical to that of the space-time wave function. The Lorentz squeeze properties of these wave functions are also the same. This aspect of the squeeze has been exhaustively discussed in the literature [9, 13, 14].

When the hadron is at rest with $\eta = 0$, both wave functions behave like those for the static bound state of quarks. As η increases, the wave functions become continuously squeezed until they become concentrated along their respective positive light-cone axes. Let us look at the z-axis projection of the space-time wave function. Indeed, the width of the quark distribution increases as the hadronic speed approaches that of the speed of light. The position of each quark appears widespread to the observer in the laboratory frame, and the quarks appear like free particles.

The momentum-energy wave function is just like the space-time wave function, as is shown in Fig. 3. The longitudinal momentum distribution becomes wide-spread as the hadronic speed approaches the velocity of light. This is in contradiction with our expectation from non-relativistic quantum mechanics that the width of the momentum distribution is inversely proportional to that of the position wave function. Our expectation is that if the quarks are free, they must have their sharply defined momenta, not a wide-spread distribution.

However, according to our Lorentz-squeezed space-time and momentum-energy wave functions, the space-time width and the momentum-energy width increase in the same direction as the hadron is boosted. This is of course an effect of Lorentz covariance. This indeed is the key to the resolution of the quark-parton paradox [9, 13].

After these qualitative arguments, we are interested in whether Lorentz-boosted bound-state wave functions in the hadronic rest frame could lead to parton distribution functions. It is thus possible to compare the oscillator-based parton distribution with that observed in high-energy laboratories [15].

Feynman's parton picture is one of the most controversial models proposed in the 20th century. The original model is valid only in Lorentz frames where the initial proton moves with infinite momentum. It is gratifying to note that this model can be produced as a limiting case of one covariant model which produces the quark model in the frame where the proton is at rest.

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