Relativistic Contents in the Poisson Bracket

Let us consider the two-dimensional phase space consisting of x and p coordinates. Then, the linear canonical transformations consists of the rotation around the origin, namely

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix},$$

and the squeeze along the x direction:

$$\begin{pmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix},$$

The rotation and squeeze are generated is generated by

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

respectively. In order to construct a set of closed set of commutation relations, we have to introduce another matrix, namely

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

These three matrices are all imaginary, and they generate transformation matrices with real elements. These transformation matrices squeeze and rotate the two dimensional objects. They are area-preserving transformations, and are called canonical transformations.

Let us introduce new notations:

$$J_2 = \frac{1}{2}\sigma_2, \quad K_3 = \frac{i}{2}\sigma_3, \quad K_i = \frac{i}{2}\sigma_1,$$

Then they satisfy the following closed set of consummation relations

$$[J_2, K_3] = iK_i, \quad [J_2, K_1] = -iK_2, \quad [K_1, K_3] = iJ_2.$$

This set of commutation relations is exactly the same as the set for the group of Lorentz transformations applicable to two space-like and one time-like directions. The generators applicable to the coordinate are given in Table **??**

Table 1: Two-by-two and four-by-four representations of the Sp(2) group. The two-by-two representation is applicable to the two-dimensional phase space of x and p. The four-by-four matrices generate Lorentz group applicable to the space of (x, y, z, t).

	Sigma	Phase Space	Minkowski Space
J_2	$\frac{1}{2}\sigma_2$	$\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
		rotation	rotation around y
K_3	$\frac{i}{2}\sigma_3$	$\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$
		squeeze	boot along z
K_1	$\frac{i}{2}\sigma_1$	$\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$
		squeeze	boost along x

In the four-by-four Minkowskian space, J_2 generates rotations around the y axis:

$$e^{-i\phi J_2} = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the K_3 and K_1 matrices lead to Lorentz boost matrices

$$e^{-i\eta K_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\eta & \sinh\eta \\ 0 & 0 & \sinh\eta & \cosh\eta \end{pmatrix},$$

and

$$e^{-i\lambda K_1} = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix},$$

along the z and x directions respectively.

These transsformations are illustrated in Fig. **??**. These transformations in the Minkowskian space leave the y axis unchanged.

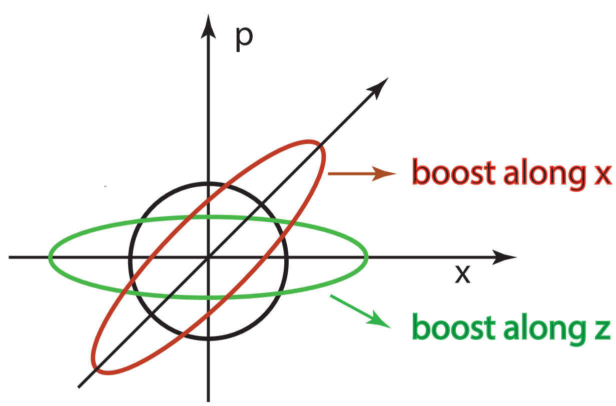

Figure 1: Rotation and squeezes in the two-dimensional space of x and P. The rotaton corresponds to the rotation around the y axis, while the squeezez correspond to the Lorentz-boosts along the z and x directions in the Minkowskian space, respectively.