This is the last homework!

Homework #13 — Phys623 — Spring 1999
Deadline: 5 p.m., Friday, May 14, 1999.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Scattering Theory (Chapter 18, see also Chapters XVII and XVIII from Landau and Lifshitz)

Scattering with Absorption

1. [**5 points**] Consider scattering on a "black disk" of radius a, which absorbs all particles for $r \leq a$ and is ineffective for $r > a$. Idealize this by taking $s_l = 0$ for $r \leq a$ and $s_l = 1$, $\delta_l = 0$ for $r > a$ in Eq. (18.38) of Schwabl.

Calculate $\sigma_{\rm el}$, $\sigma_{\rm inel}$, $\sigma_{\rm tot}$ and check whether the optical theorem (18.44) is satisfied. In the calculation, assume that the energy of the particles is high, so that $ka \gg 1$, and the sum over l can be replaced by an integral. Explain your result for $\sigma_{\rm el}$ and $\sigma_{\rm inel}$ qualitatively following Sec. 18.10.

2. [**5 points**] As discussed in Problem 2 of Homework 12, the scattering of slow (low energy) particles is characterized by a scattering length a_0. In the presence of absorption, the scattering length a_0 is complex and has a negative imaginary part (see Sec. 18.6). Here my definition of the complex phase $\delta_0 = -ka_0$ includes both Schwabl's real variables s_0 and δ_0.

Find the behavior of $\sigma_{\rm el}$ and $\sigma_{\rm inel}$, in the limit of small energy of the scattering particle in terms of the complex scattering length a_0 (**see Hints**). Which of the two cross sections dominates in that limit? [*This is the so-called* $1/v$ *law due to Hans Bethe.*]

Scattering of Fast Particles on Atoms

3. *Adapted from Qualifier, January 1998, Fall 1984, Fall 1983, Spring 1981, II-3, §139 of Landau-Lifshitz.*

A charged particle is scattered elastically on the electric potential $\varphi(\vec{r})$ produced by an atom, which has Z electrons and Z protons in the nucleus.

- **(a)** [**3 points**] In the Born approximation, find a general expression for the differential cross section of scattering in terms of a given charge distribution of electrons $n(\vec{r})$. The nucleus is treated as a point charge Ze (**see Hints**).
  [*The following quantity* $F(\vec{q}) = \int n(\vec{r}) e^{-i\vec{q}\vec{r}} d^3r$ *is called the* atomic form-factor.]
- **(b)** [**3 points**] How does the answer to question 3a simplify when $qa \gg 1$, where a is the width of the electron charge distribution in the atom?
- **(c)** [**3 points**] Using your solution of Problem 3a, find a general expression for the differential cross section of scattering in the forward direction ($\theta = 0$) in terms of $n(\vec{r})$. [**5 points**]
- **(d)** [**3 points**] Using your solution of Problem 3a, calculate differential and total cross sections of scattering on the hydrogen atom in the ground state. When calculating the latter quantity, assume that the energy of the scattering particle is sufficiently high.

4. *Adapted from Qualifier, Fall 1996, II-3, §148 of Landau-Lifshitz.*

Consider *inelastic* scattering of fast charged particles on an atom that has Z electrons and Z protons in the nucleus. The scattering particles and the atom interact via the Coulomb potential. Neglect recoil of the atom.

(a) [**7 points**] Find a general expression for the differential cross section of scattering with the excitation of the atom from the state $|0\rangle$ (say, ground state) to the state $|n\rangle$, where n denotes all quantum numbers of the excited state.

Assume that the scattering particles have high velocity and use an approximation analogous to the Born approximation, that is use the Fermi Golden Rule to calculate the scattering rate. [*In this context the approximation was actually developed by Hans Bethe.*]

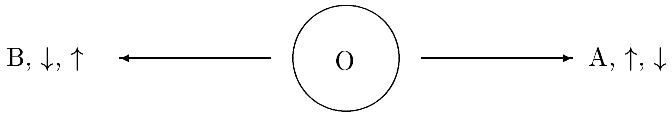
Make sure that your result agrees with your solution of Problem 3a in the case where the final state of the atom $|n\rangle$ coincides with its initial state $|0\rangle$, that is the scattering is elastic.

(b) [**7 points**] Using your solution of Problem 4a, calculate differential and total cross sections of scattering on the hydrogen atom with its excitation from the 1s state to the 2s state. When calculating the latter quantity, assume that the velocity of the scattering particle is sufficiently high.

Compare the total cross section of the 1s→2s scattering with the total cross section of the 1s→1s scattering found in Problem 3a.

The Density Matrix (Chapter 20.2)

5. [**3 points**] A spinless particle O decays into two spin-1/2 particles, A and B, that fly in the opposite directions.



The spin-zero wave function of the system after the decay is

$$|\psi(s_A, s_B)\rangle = \frac{|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B}{\sqrt{2}}, \tag{1}$$

where s_A and s_B are the spin variable of the particles A and B taking the values ↑ and ↓. Since the system is in a pure state, the density matrix is given by Eq. (20.2):

$$\rho(s_A, s_B, s'_A, s'_B) = |\psi(s_A, s_B)\rangle\langle\psi(s'_A, s'_B)|. \tag{2}$$

We use the beam of particles A for our experiments in scattering, and we don't care what happen to particles B. (They hit the wall of the reactor and get absorbed.) So, for description of our experiments we need only the *reduced* density matrix ρ_r, which is obtained by taking trace over the coordinates of the particle B and depends only on the coordinates of the particle A:

$$\rho_r(s_A, s'_A) = \mathrm{Tr}_B\rho = \sum_{s_B} \rho(s_A, s_B, s'_A, s_B). \tag{3}$$

Calculate ρ_r using Eqs. (3), (2), and (1) and show that ρ_r is the density matrix of the spin-unpolarized state (Eq. (20.37) with $\mathbf{b}$=0).

Hints

2 Substitute $e^{2i\delta_0} = e^{-2ika_0}$ into Eqs. (18.38), (18.39), and (18.41) and take the limit where k is small.

3a Use the Coulomb law to relate the density of charge and the electric potential $\varphi(\vec{r})$. Also,

$$\int \frac{e^{-i\vec{q}\vec{r}}}{|\vec{r} - \vec{r}_a|}\, d^3\vec{r} = \frac{4\pi}{q^2}\, e^{-i\vec{q}\vec{r}_a}. \tag{4}$$