Homework #9 — Phys622 — Fall 1998
Deadline: Friday, November 6, 1998.
Turn in homework in the class or put in
the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Motion in an Electromagnetic Field (Chapter 7)

1. Consider the 2D isotropic harmonic oscillator, whose Hamiltonian is

$$\hat{H}_{\rm 2Dosc} = \frac{1}{2\mu}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{\mu\omega^2}{2}\left(x^2 + y^2\right). \tag{1}$$

We denote the mass of a particle by the letter μ to distinguish from the eigenvalue m of $\hat{L}_z$.

(a) [**3 points**] Write the energy eigenvalues of Hamiltonian (1) in the Cartesian coordinates x and y. Determine the degeneracy of the energy levels and the parity of the eigenstates.

Now, let us study Hamiltonian (1) using the radial and angular coordinates in 2D.

(b) [**5 points**] Let us introduce the dimensionless variables $\varepsilon = E/\hbar\omega$ and $\xi = \rho\sqrt{\mu\omega/\hbar}$.
Argue that the radial Schrödinger equation (Eq. (11) of Homework 6) for the 2D oscillator is the same as for the 3D oscillator with the replacement $l \to |m| - 1/2$. Using your solution of Problem 7 of Homework 7, argue that the radial wave function can be written as:

$$R_{Em} = \xi^{|m|} e^{-\xi^2/2} w(\xi), \tag{2}$$

where $w(\xi)$ is given by a power series

$$w(\xi) = \sum_{r=0}^{s_{\rm max}} C_s \xi^s, \tag{3}$$

which truncates at $s_{\rm max} = 2N$. That gives the energy eigenvalues

$$E = (2N + |m| + 1)\hbar\omega, \tag{4}$$

where $N = 0, 1, 2, \ldots$ is the radial quantum number, or

$$E = (n+1)\hbar\omega, \tag{5}$$

where $n = 2N + |m| = 0, 1, 2, \ldots$ is the principal quantum number.

(c) [**3 points**] Represent the energy levels as points on a diagram with $|m|$ on the horizontal axis and E on the vertical axis. Connect the points with equal energies. For a given value of n, what are the allowed values of $|m|$? Find the degeneracy and the parity of the n-th eigenstate and compare the result with that of Problem 1a.

2. Consider a particle of mass μ in 2D in a homogeneous magnetic field $\mathbf{B}$ perpendicular to the (x, y) plane. Choose the symmetric gauge to describe the magnetic field:

$$\mathbf{A} = \frac{1}{2}\,\mathbf{B} \times \mathbf{r}. \tag{6}$$

(a) [**3 points**] Show that the Hamiltonian in the magnetic field,

$$\hat{H}_{\rm mag} = \frac{(\hat{\mathbf{p}} - e\mathbf{B}\times\mathbf{r}/2c)^2}{2\mu}, \tag{7}$$

can be written as

$$\hat{H}_{\rm mag} = \hat{H}_{\rm 2Dosc}(\omega_c/2, \mu) - \hat{L}_z\omega_c/2, \tag{8}$$

in terms of the Hamiltonian (1) of the 2D harmonic oscillator with the frequency $\omega_c/2$ and mass μ, where $\omega_c = eB/c\mu$ is the cyclotron frequency. (See Sec. 7.2.)

(b) [**3 points**] Argue that the eigenstates of $\hat{H}_{\rm 2Dosc}$ with the quantum numbers N and m, found in Problem 1, are also the eigenstates of $\hat{H}_{\rm mag}$ and find the eigenenergies of $\hat{H}_{\rm mag}$ in terms of N and m. Represent the integers $N \geq 0$ and $m \in [-\infty, +\infty]$ as points on the (m, N) plane and connect the points that produce the same energy of $\hat{H}_{\rm mag}$. Show that eigenenergies of $\hat{H}_{\rm mag}$ can be written as $E = (n + 1/2)\hbar\omega_c$, in agreement with Eq. (7.37), and discuss their degeneracy.

(c) [**5 points**] Instead of the two coordinates, x and y, let us introduce the complex variable $z = x+iy$. Using the solutions of Problems 2b and 1, show that the wave functions of the lowest Landau level can be written as

$$\psi_{0,m}(x,y) = z^m \exp\left(-\frac{eB}{4\hbar c}|z|^2\right), \qquad m \geq 0. \tag{9}$$

(Functions (9) are not normalized.)

Argue that a generic wave function of the lowest Landau level has the form:

$$\psi_0(x,y) = f(z) \exp\left(-\frac{eB}{4\hbar c}|z|^2\right), \tag{10}$$

where $f(z)$ is an *analytic function* of the variable z; that is, a function that contains any powers of z in its Taylor expansion, but no powers of z^*:

$$f(z) = \sum_{m=0}^{\infty} C_m z^m. \tag{11}$$

Give an example of a wave function of the first excited Landau level and show that this function does not belong to the class (9).

Instead of the Taylor expansion coefficients C_n, the analytic function $f(z)$ (10) can be also characterized by the positions z_k of its zeros:

$$f(z) = \prod_k (z - z_k). \tag{12}$$

How does the phase of the wave function (12) change as we go around a particular nondegenerate zero z_k in the (x, y) plane?

(d) [**3 points**] Show that, for a large value of m, the probability density of the wave function $\psi_{0,m}$ (9) is concentrated at the radius $r_m = \sqrt{2m}l$, where $l = \sqrt{c\hbar/eB}$ is called the *magnetic length.*

(e) [**3 points**] Suppose the system has the shape of a 2D disk of a big radius R. In this case, the biggest possible value of m is restricted roughly by the condition that $r_m < R$. Calculate the number N of the lowest Landau level states that fit into the disk. Show that the number of the states per unit area is the same as the number found in Problem 6c of Homework 8.

3. [**7 points**] *Adapted from Physics Qualifier, September 1970, Problem II-1. This Problem is relevant for the so-called* quantum dot *in a magnetic field, a popular subject in semiconductor physics now.*

Let us combine Problems 1 and 2. Let us consider an isotropic 2D oscillator with the frequency ω in a magnetic field applied perpendicular to the plane of the oscillator, such that the cyclotron frequency of the field in the absence of the oscillator would be ω_c.

Find the energy levels of the system. How do the energy levels of the 2D oscillator split, when a weak magnetic field is applied? How do the Landau energy levels split, when a weak oscillator potential is applied?

Operators, Matrices, State Vectors (Chapter 8)

4. [**4 points**] Schwabl's Problem 8.5.

5. Let us consider the Heisenberg representation of a Hermitian operator A_H that does not depend on time explicitly: $\partial A/\partial t = 0$. You may think of the coordinate operator $\mathbf{r}$ as an example of such an operator A_H.

(a) [**2 points**] Show that

$$\left\langle m \left| \frac{dA_H}{dt} \right| n \right\rangle = \frac{i}{\hbar}(E_m - E_n)\langle m|A_H|n\rangle, \tag{13}$$

where dA_H/dt is the time derivative of A_H, and the matrix elements are taken with respect to the eigenstates $|m\rangle$ and $|n\rangle$ of the Hamiltonian with the eigenenergies E_m and E_n.

(b) [**2 points**] Show that if the average value an operator A_H in a discrete energy eigenstate $|n\rangle$ is finite, then the average value of its time derivative vanishes:

$$\left\langle n \left| \frac{dA_H}{dt} \right| n \right\rangle = 0. \tag{14}$$

This statement is also true in classical mechanics. (Coordinate $\mathbf{r}$ and velocity $\mathbf{v} = \dot{\mathbf{r}}$ in a bound state give an illustration of this statement.)