Homework #8 — Phys622 — Fall 1998
Deadline: Friday, October 30, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Motion in an Electromagnetic Field (Chapter 7)

1. [**5 points**] Schwabl's Problem 7.1.

2. [**5 points**] Schwabl's Problem 7.4. Do only the part (b) of this problem.

By calculating $\oint d\mathbf{r}\,\mathbf{A}(r)$, show that the magnetic flux Φ through the rotator is equal to $\pi r_0^2 B$.

Plot the ground state energy of the rotator as a function of Φ and show that this dependence is periodic with the period $2\Phi_0$ (7.27).

3. Consider a particle in two dimensions, x and y, in a constant homogeneous magnetic field $\mathbf{B}$ perpendicular to the (x,y) plane and a homogeneous, perhaps time-dependent, electric field $\mathbf{E}$ parallel to the (x,y) plane. Let us introduce the operator of velocity

$$\hat{\mathbf{v}} = \frac{1}{m}\left(\hat{\mathbf{p}} - \frac{e\mathbf{A}}{c}\right), \tag{1}$$

proportional to the kinetic momentum (7.11).

(a) [**5 points**] Using the Ehrenfest theorem (Sec. 2.6), Hamiltonian (7.2), and the commutator (7.30), derive equations of motion for the average coordinate $\langle\mathbf{r}\rangle$ and the average velocity $\langle\mathbf{v}\rangle$. **(see Hints)**

Compare the equation of motion for $\langle\mathbf{v}\rangle$ with the classical equation (B.3):

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right). \tag{2}$$

(b) [**5 points**] Suppose the electric field is periodic in time: $\mathbf{E}(t) = \mathbf{E}(\omega)e^{-i\omega t}$, where $\omega > 0$. (Well, more precisely, $\mathbf{E}(t) = \operatorname{Re}\mathbf{E}(\omega)e^{-i\omega t}$.) We are looking for periodic solutions of Eq. (2): $\mathbf{v}(t) = \mathbf{v}(\omega)e^{-i\omega t}$. By substituting this formula into Eq. (2), express $v_\pm(\omega) = v_x(\omega) \pm iv_y(\omega)$ in terms of $E_\pm(\omega) = E_x(\omega) \pm iE_y(\omega)$.

Is there a frequency ω_c such that the velocity $\mathbf{v}(\omega) \to \infty$ when $\omega \to \omega_c$? What is the physical origin of this effect? Compare that frequency with the cyclotron frequency (7.36).

Suppose, the electric field is right- or left-circularly polarized: $E_y(\omega) = \pm iE_x(\omega)$. Does the cyclotron resonance take place for both right and left polarizations of the electric field for a given orientation of the magnetic field?

(c) [**3 points**] Taking the limit of $\omega = 0$ in the equations of Problem 3b, find the magnitude and orientation of a stationary velocity $\langle\mathbf{v}\rangle$ in the case, where the electric field is time-independent.

4. (a) [**3 points**] Different gauges may describe the same homogeneous magnetic field $\mathbf{B}$. One possible choice is the so-called *symmetric* gauge:

$$\mathbf{A} = \frac{1}{2}\,\mathbf{B}\times\mathbf{r}. \tag{3}$$

Another valid choice of the gauge is a displaced symmetric gauge:

$$\mathbf{A}' = \frac{1}{2}\,\mathbf{B}\times(\mathbf{r}+\mathbf{w}), \tag{4}$$

where $\mathbf{w}$ is a constant two-dimensional vector perpendicular to $\mathbf{B}$. Check that both vector potentials (3) and (4) produce the same magnetic field according to Eq. (7.1).
Show that the difference between the vector potentials,

$$\mathbf{A}' - \mathbf{A} = \frac{1}{2}\,\mathbf{B} \times \mathbf{w} = \frac{\partial \Lambda(\mathbf{r})}{\partial \mathbf{r}}, \tag{5}$$

can be represented as the gradient of a function $\Lambda(\mathbf{r})$ and find $\Lambda(\mathbf{r})$.

(b) [**5 points**] Suppose, that $\psi_0(\mathbf{r},t)$ is a solution of the Schrödinger equation in the homogeneous magnetic field:

$$i\hbar\frac{\partial \psi_0(\mathbf{r},t)}{\partial t} = \frac{(\hat{\mathbf{p}} - e\mathbf{B}\times\mathbf{r}/2c)^2}{2m}\psi_0(\mathbf{r},t). \tag{6}$$

Show that the wave function, displaced by a vector $\mathbf{w}$,

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r}-\mathbf{w},t)e^{i\varphi(\mathbf{r},\mathbf{w})}, \tag{7}$$

where $\varphi(\mathbf{r},\mathbf{w})$ is some phase, is also a solution of the same Schrödinger equation (6). Find the phase $\varphi(\mathbf{r},\mathbf{w})$. What is the relation between the phase $\varphi(\mathbf{r},\mathbf{w})$ and the function $\Lambda(\mathbf{r})$ found in Problem 4a?

5. *Adapted from Physics Qualifier, January 1991, Problem II-5.*

Let us consider a two-dimensional particle in the homogeneous magnetic field in the so-called *Landau gauge*:

$$A_y = Bx, \qquad A_x = 0. \tag{8}$$

(a) [**5 points**] By writing explicitly the Schrödinger equation and using analogy with a one-dimensional oscillator, find the energy eigenvalues and eigenfunctions. Is there any degeneracy of the eigenstates? Do the energy levels agree with Eq. (7.37)?
Sketch schematically an eigenfunction $\psi^{(0)}(x,y)$ of the lowest energy level. What can you say about localization of the wave function in the x and y directions?

(b) [**5 points**] You should have found in Problem 5a that the energy eigenfunctions are degenerate in p_y, the eigenvalue of the momentum in the y direction. Any superposition of degenerate eigenstates is also an eigenstate. Thus, we may transform a set of degenerate eigenstate $\psi_{p_y}^{(n)}(x,y)$, where n is the number of the Landau level, into another set of degenerate eigenstates labeled by an index Y:

$$\psi_Y^{(n)}(x,y) = \int e^{ip_yY/\hbar}\psi_{p_y}^{(n)}(x,y)dp_y. \tag{9}$$

Perform the transformation (9) explicitly for the wave functions of the lowest Landau level ($n=0$). What can you say about the localization of a wave function $\psi_Y^{(0)}(x,y)$ in the x and y directions?

(c) [**5 points**] Suppose, the size of the system is limited by the width L_y in the y direction and the length L_x in the x direction. Both sizes are much greater than a natural characteristic width of a wave function.
Calculate the degeneracy of each energy eigenvalue. To simplify the problem, assume that the periodic boundary condition is imposed in the y direction:

$$\psi(x,y) = \psi(x,y+L_y). \tag{10}$$

Condition (10) permits only discrete values of p_y, which labels the eigenstates found in Problem 5a. Count the number of available values of p_y to find the degeneracy. **(see Hints)**

(d) [**3 points**] Suppose, each eigenstate can be occupied by only one electron. (The electrons are fermions, and let us neglect the spin for simplicity.) Suppose the electrons completely fill N (lowest) Landau levels.
Using the solution of Problem 5c, calculate the concentration of the electrons per unit area of the (x,y) space. Calculate the magnetic flux per one electron and compare the result with the quantum of magnetic flux $2\Phi_0$ (7.27).

6. *Adapted from Physics Qualifier, January 1991, Problem II-5.*

Now, let us consider Problem 5 in the presence of a constant electric field E_x along the x direction, $|E_x| < |B|$.

(a) [**5 points**] By writing explicitly the Schrödinger equation and using analogy with a one-dimensional oscillator, find the energy eigenvalues and eigenfunctions of Hamiltonian (7.2) with the vector potential (8) and scalar potential $\phi = -E_x x$. Is there any degeneracy of the eigenstates? Compare the eigenstates with those found in Problem 5a.

(b) [**5 points**] Calculate the total electric current $\mathbf{J}$ for an eigenstate found in Problem 6a. To obtain the total current, integrate the current density $\mathbf{j}$, given in Problem 7.1 of Schwabl times the electron charge e, across the direction of the current.

What is the orientation of the current relative to the direction of the electric field? Compare your result with the solution of Problem 3c.

[*The occurrence of electric current in the direction perpendicular to both magnetic and electric fields is called the* Hall effect. *It was discovered by E. H. Hall in 1879 at the Johns Hopkins University in Baltimore.*]

(c) [**5 points**] Suppose the two-dimensional system is filled with electrons at the concentration of n electrons per unit area. Calculate the Hall resistance, defined as the ratio of the voltage V_x in the x direction to the current J_y in the y direction: **(see Hints)**

$$R_H = \frac{V_x}{J_y}. \tag{11}$$

Suppose that the electron concentration is such that N Landau levels are completely filled (see Problem 5d). Calculate the Hall resistance in this case and compare the result with the *quantum of resistance*:

$$R_0 = \frac{h}{e^2} = 25.8\,\mathrm{k}\Omega. \tag{12}$$

Hints

3a Don't forget that the operator $\hat{\mathbf{v}} = (\hat{\mathbf{p}} - e\mathbf{A}/c)/m$ may have an explicit dependence on time through $\mathbf{A}$.

5c Possible values of p_y are restricted to a finite range, because p_y determines the position of a wave function in x direction, and the size of the system is limited in the x direction by L_x.

6c To calculate the total electric current, multiply the current carried by one quantum state (see Problem 6b) by the total number of electrons.