Homework #7 — Phys622 — Fall 1998
Deadline: Wednesday, October 21, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

Victor Yakovenko, Assistant Professor
Office: Physics 2314
Phone: (301)–405–6151
E-mail: yakovenk@physics.umd.edu

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

The Central Potential (Chapter 6)

1. [**5 points**] Schwabl's Problem 6.5.

2. [**5 points**] Schwabl's Problem 6.6.

3. [**5 points**] Schwabl's Problem 6.7. (*Physics Qualifier, August 1995, 1998, Problem II-1.*)

4. [**7 points**] *Adapted from Physics Qualifier, January 1983, Problem II-1.*

A particle of mass m experiences the following three-dimensional potential:

$$V(r) = -\frac{\lambda_1}{r} + \frac{\lambda_2}{r^2}, \tag{1}$$

where $\lambda_1, \lambda_2 > 0$.

Find the discrete energy spectrum and compare its degeneracy with that of the hydrogen atom. Write the wave functions of the two lowest energy levels for a given angular momentum l (do not bother to normalize).

5. **(a)** [**5 points**] Suppose a Hamiltonian depends on a parameter λ. Then, the eigenfunctions and eigenvalues also depend on λ:

$$\hat{H}(\lambda)\psi_n(\lambda) = E_n(\lambda)\psi_n(\lambda), \tag{2}$$

where the parameter n labels different eigenstates, and the eigenfunctions ψ_n are normalized. Show that

$$\frac{\partial E_n(\lambda)}{\partial \lambda} = \left\langle \frac{\partial \hat{H}(\lambda)}{\partial \lambda} \right\rangle_n, \tag{3}$$

where the average is taken with respect to $\psi_n(\lambda)$. **(see Hints)**

Eq. (3) is valid not only for a Hamiltonian, but for any Hermitian operator and its eigenvalues and eigenfunctions.

(b) [**3 points**] Let us consider the discrete spectrum of the radial Schrödinger equation (6.11). For a given l, we label the energy levels consecutively in the order of increase by an integer N (the radial quantum number): E_{Nl}. By this definition, E_{Nl} increases with increasing N at a fixed l. How does the energy E_{Nl} change with increasing l at a fixed N? **(see Hints)**

6. A particle of zero spin is subject to a spherically-symmetrical potential. Answer the following questions and compare results with the cases of hydrogen atom and the potential studied in Problem 4. **(see Hints)**

(a) [**3 points**] May energy levels of the particle have the degeneracy $g = 2$?

(b) [**3 points**] What are the possible values of degeneracy g for the first excited energy level?

(c) [**3 points**] A quantum level has the degeneracy $g = 9$. What are the possible values of the angular momentum l of this level?

Let us label the energy levels in the order of increase by an integer n (the principal quantum number) starting from $n = 1$ regardless of N and l: E_n.

(d) [**3 points**] What is the maximal possible value of the angular momentum l in a state ψ_n?

(e) [**3 points**] What is the maximal possible degeneracy g of a state ψ_n?

7. Consider the 3D isotropic harmonic oscillator, whose Hamiltonian is

$$\hat{H} = \frac{1}{2\mu}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right) + \frac{\mu\omega^2}{2}\left(x^2 + y^2 + z^2\right). \tag{4}$$

We denote the mass of a particle by the letter μ to distinguish from the eigenvalue m of $\hat{L}_z$.

(a) [**5 points**] Write the energy eigenvalues and eigenfunctions of Hamiltonian (4) in the Cartesian coordinates x, y and z in terms of the eigenfunctions of a 1D harmonic oscillator. Determine the degeneracy of the energy levels and the parity of the eigenstates.

Now, let us study the Hamiltonian (4) using the radial and angular coordinates in 3D.

(b) [**5 points**] Introduce the dimensionless variables $\varepsilon = E/\hbar\omega$ and $\xi = r\sqrt{\mu\omega/\hbar}$. Represent the radial wave function in terms of a new function $w(\xi)$:

$$u_{El}(r) = \xi^{l+1} e^{-\xi^2/2} w(\xi). \tag{5}$$

By substituting Eq. (5) into the radial Schrödinger equation, find a differential equation for $w(\xi)$.

(c) [**5 points**] Represent $w(\xi)$ as a power series

$$w(\xi) = \sum_{s=0}^{\infty} C_s \xi^s \tag{6}$$

and find a recursion relation between the coefficients of expansion C_s.
Argue that the series (6) must terminate at some finite s, if the behavior of the solution at $\xi \to \infty$ is to be acceptable. Argue also that the maximal value $s_{\max} = 2N$ is necessarily even. Using these conditions, show that the energy eigenvalues are

$$E = (2N + l + 3/2)\hbar\omega, \tag{7}$$

where $N = 0, 1, 2, \ldots$ is the radial quantum number, or

$$E = (n + 3/2)\hbar\omega, \tag{8}$$

where $n = 2N + l = 0, 1, 2, \ldots$ is the principal quantum number.

(d) [**5 points**] Represent the energy levels as points on a diagram with l on the horizontal axis and E on the vertical axis. Connect the points with equal energies. For a given value of n, what are the allowed values of l? Find the degeneracy and the parity of the n-th eigenstate and compare the result with that of Problem 7a.

(e) [**3 points**] Write explicitly all eigenfunctions corresponding to $n = 0$ and 1. Represent them as superpositions of the corresponding eigenfunctions found in Problem 7a in the Cartesian coordinates.

Hints

5a Differentiate Eq. (2) by λ and take a scalar product with $\psi_n(\lambda)$. Take into account that $\psi_n(\lambda)$ is normalized.

5b Use the solution of Problem 5a treating l as the parameter λ.

6 Use the solution of Problem 5b.