Homework #5 — Phys622 — Fall 1998
Deadline: Wednesday, October 7, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

Victor Yakovenko, Assistant Professor
Office: Physics 2314
Phone: (301)–405–6151
E-mail: yakovenk@physics.umd.edu

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Angular Momentum (Chapter 5)

Use only commutation relations to solve Problems 1–4; do not use explicit form of $Y_{lm}(\vartheta,\varphi)$. This will make the results valid not only for the orbital (integer), but also for the spin (half-integer) angular momentum.

1. **[3 points each part]** Schwabl's Problem 5.1.

   In part (b), calculate $(\Delta\hat{L}_i)^2$ for a general state ψ_{lm}, not only $m=l$. Obtain the uncertainty relation
   $$\Delta\hat{L}_x\Delta\hat{L}_y \geq \frac{\hbar}{2}\,|\langle\hat{L}_z\rangle| \tag{1}$$
   by substituting $\hat{L}_x$ and $\hat{L}_y$ into Eq. (4.8) instead of $\hat{H}_1$ and $\hat{H}_2$. Check whether the directly calculated $(\Delta\hat{L}_i)^2$ satisfy the inequality (1).

   In part (c) show also that the uncertainty relation (1) is saturated to equality in the states with $m=\pm l$.

2. **[3 points]** Schwabl's Problem 5.3 (Use the Schwarz inequality (4.1)).

3. **[3 points]** Schwabl's Problem 5.4 (Use Eq. (5.2a)).

4. **[5 points]** Show that if an operator $\hat{A}$ commutes with the angular momentum: $[\hat{L}_i,\hat{A}]=0$ (in other words, $\hat{A}$ is a scalar), the matrix elements
   $$\int d^3r\,\psi^*_{nlm}\hat{A}\psi_{nlm'} \tag{2}$$
   do not vanish only when $m=m'$ and do not depend on m. In Eq. (2), n denotes quantum numbers other than l and m.

The following problems 5 and 6 are specific to the cases of $l=1$ and $l=2$, respectively. Use the explicit forms of $Y_{1m}(\vartheta,\varphi)$ and $Y_{2m}(\vartheta,\varphi)$ from Eq. (5.34) in these problems.

5. **[3 points each part]**

   (a) Show that the functions
   $$\sqrt{\frac{3}{4\pi}}\frac{x}{r},\qquad \sqrt{\frac{3}{4\pi}}\frac{y}{r},\qquad \sqrt{\frac{3}{4\pi}}\frac{z}{r} \tag{3}$$
   are the eigenfunctions of $\hat{\mathbf{L}}^2$ with $l=1$ and the eigenfunctions of, correspondingly, $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ with zero eigenvalues. **(see Hints)**

   (b) Expand the functions (3) over the functions Y_{11}, Y_{10}, and Y_{1-1} given by Eq. (5.34) and vice versa. Argue that the functions (3) form a complete orthonormal set of functions in the angular space with $l=1$.

   (c) A particle has the wave function $\psi(x,y,z)=N(x+y+2z)e^{-r/a}$, where N is a normalization constant. Calculate the probabilities to find the particle in the eigenstates ψ_{lm} (which are defined relative to the z axis).

(d) In the case $l = 1$, find the wave function $\psi_{\alpha,\beta}(\vartheta, \varphi)$ of the state where the projection of the angular momentum on a given direction $\mathbf{n}$ is zero: $\mathbf{n} \cdot \hat{\mathbf{L}}\,\psi = 0$. The direction $\mathbf{n}$ is characterized by the polar angle α and the azimuthal angle β. Show the explicit dependence of $\psi_{\alpha,\beta}(\vartheta, \varphi)$ on ϑ and φ, as well as α and β. **(see Hints)**

6. [**5 points**] By expressing the functions (4) in terms of the functions (5.34), show that the following set of wave functions:

$$\sqrt{\frac{5}{16\pi}}\frac{2z^2 - x^2 - y^2}{r^2}, \qquad \sqrt{\frac{5}{16\pi}}\frac{zx}{r^2}, \qquad \sqrt{\frac{5}{16\pi}}\frac{zy}{r^2}, \qquad \sqrt{\frac{5}{16\pi}}\frac{xy}{r^2}, \qquad \sqrt{\frac{5}{64\pi}}\frac{x^2 - y^2}{r^2}, \tag{4}$$

forms a basis for the states with $l = 2$. Sketch the spatial shapes of the wave functions (4) with the signs of the wave function indicated.

The wave functions (3) and (4) are frequently used to describe electron orbitals in crystals. The d-electrons ($l = 2$) of copper (Cu) belonging to the $x^2 - y^2$ orbital (the last one in Eq. (4)) play crucial role in high-temperature superconductivity observed in copper oxides.

Hints

5a First, prove the statement for $\hat{L}_z$ by direct comparison with Eq. (5.34). Prove the rest by permutation of the x, y, and z axes.

5d When $\mathbf{n}$ is directed along the z-axis, the required wave function $\psi_{0,0}(\vartheta, \varphi)$ is known and can be written as $\sqrt{3/4\pi}\,\mathbf{n} \cdot \mathbf{r}/r$. What would the function be when $\mathbf{n}$ points anywhere?