Homework #4 — Phys622 — Fall 1998
Deadline: Friday, October 2, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

One-Dimensional Problems, Symmetry (Chapter 3)

1. [**5 points**] Schwabl's Problem 3.17.

   Also, prove formally that the eigenfunctions with the energies $0 < E < V_1$ and $E < 0$ are not degenerate. Show first that if the Schrödinger equation has two solutions $\psi_1(x)$ and $\psi_2(x)$ of the same energy, the following combination (the Wronskian determinant) is independent of x:

$$\psi_1(x)\frac{d\psi_2(x)}{dx} - \psi_2(x)\frac{d\psi_1(x)}{dx} = \text{const.} \tag{1}$$

   Then use Eq. (1) at $x \to +\infty$ or $x \to -\infty$, where the wave functions must vanish, to prove the theorem.

2. [**5 points**] Schwabl's Problem 3.16.

3. **(a)** [**1 point**] In classical physics, both the coordinate x and the momentum $m\,dx/dt$ reverse signs upon reflection. Show that this is also true in quantum mechanics:

$$\hat{P}\hat{x} = -\hat{x}\hat{P}, \qquad \hat{P}\hat{p} = -\hat{p}\hat{P}, \tag{2}$$

   where $\hat{P}$, $\hat{x}$, and $\hat{p}$ are the parity, coordinate, and momentum operators. What is the relation between the probability current densities (2.59) calculated for the wave functions $\psi(x)$ and $\psi(-x)$?

   (b) [**3 points**] Let us consider a Gaussian packet whose initial wave function $\psi(x,t)$ at $t = 0$ is obtained from the wave function (2.5), which we denote as $\psi_0(x,t)$, at $t = 0$ by reflection around the point $x = l$: $\psi(x,0) = \psi_0(2l - x, 0)$. Sketch and write analytically (in terms of $\psi_0(x,t)$) the wave function $\psi(x,t)$ at later moments of time. What is the initial position of this packet? Does it move to the right or to the left? Does the width of the packet increase or decrease?

   (c) [**3 points**] *Adapted from Physics Qualifier, Spring 1972, Problem II-4.*
   Consider the time evolution in one dimension of a Gaussian wave packet shown in Fig. 2.3 of Schwabl. At $t = 0$, the packet is centered at $x = 0$ and moves to the right. Suppose there is an impenetrable wall at the point $x = l$ far away from the initial position of the packet: $l \gg d$, where d is the initial width of the packet. In other words, the particle moves in the potential

$$V(x) = \begin{cases} 0, & x < l, \\ \infty, & x \geq l. \end{cases} \tag{3}$$

   Sketch the time evolution of the Gaussian wave packet in the presence of the wall. Find the time-dependent wave function $\psi(x,t)$ in this case analytically in term of the wave function $\psi_0(x,t)$ (2.8). You do not need to do any calculations in this problem. Just use symmetry principles (**see Hints**).

4. We have discussed the coordinate x reversal operation. Now let us discuss the operation of time reversal $t \to -t$. In classical physics, the coordinate x does not and the momentum $m\,dx/dt$ does change sign upon the time reversal. The Newton equations of motion typically are invariant with respect to the time reversal, that is the same equations of motion describe the evolution of the system forward in time, to the future, and backward in time, to the past. Thus, if at some moment of time we apply the time reversal operation, that is reverse the momenta, and let the system evolve with these new initial conditions, it will retrace its previous trajectory backwards. This is not true in the presence

of an external magnetic field, which is not time-reversal invariant. Unless we change the sign of the magnetic field, the system will not retrace its past trajectory if the velocity is reversed (because of the Lorentz force).

In quantum mechanics, the Schrödinger equation for the reversed time $t' = -t$:

$$-i\hbar\frac{\partial\psi(t')}{\partial t'} = \hat{H}(t')\psi(t'), \tag{4}$$

differs from the original equation

$$i\hbar\frac{\partial\psi(t)}{\partial t} = \hat{H}(t)\psi(t), \tag{5}$$

only in the sign of the time derivative. Eq. (4) can be formally cast into the form (5) by complex conjugation:

$$i\hbar\frac{\partial\psi^*(t')}{\partial t'} = \hat{H}^*(t')\psi^*(t'). \tag{6}$$

Thus we can say that complex conjugation represents the time-reversal operation in quantum mechanics. If the system is time-reversal invariant, the Hamiltonians in Eqs. (5) and (6) must coincide; thus the Hamiltonian of the system must be time-independent and *real*:

$$\hat{H}^* = \hat{H}. \tag{7}$$

This is not true in magnetic field, where the Hamiltonian is complex (see Chapter 7). Correspondingly, the complex conjugation of the Hamiltonian is equivalent to reversing the sign of the magnetic field.

(a) [**1 point**] Check how the quantum-mechanical operators $\hat{x}$ and $\hat{p} = -i\hbar\partial/\partial x$ behave upon the complex conjugation. Compare with the behavior of the corresponding classical operators upon the time reversal. What is the relation between the probability current densities (2.59) calculated for the wave functions $\psi(x)$ and $\psi^*(x)$?

(b) [**3 points**] Let us consider a Gaussian packet whose initial wave function $\psi(x,t)$ at $t=0$ is obtained from the wave function (2.5), which we denote as $\psi_0(x,t)$, at $t=0$ by complex conjugation: $\psi(x,0) = \psi_0^*(x,0)$. Sketch and write analytically (in terms of $\psi_0(x,t)$) the wave function $\psi(x,t)$ at later moments of time. What is the initial position of this packet? Does it move to the right or to the left? Does the width of the packet increase or decrease?

5. Let us consider transmission through an arbitrary potential $V(x)$ located within $|x| < a$, so that $V(x) = 0$ at $|x| > a$. A wave function of the energy $E = \hbar^2k^2/2$ has the following forms outside of the potential:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -a, \\ Fe^{ikx} + Ge^{-ikx}, & x > a, \end{cases} \tag{8}$$

The coefficients A, B, F, and G are related by the scattering matrix S and by the transfer matrix M (see Problem 4 of Homework 3). In the present Problem we discuss general limitations on the structure of these matrices imposed by symmetry.

In each of the following questions, after proving the answer in general case, verify that the corresponding requirement is satisfied by the scattering and transfer matrices of the delta-function potential, which were found in Problem 5 of Homework 3.

(a) [**3 points**] Taking into account conservation of probability, show that the matrix S must be *unitary* (**see Hints**):

$$\mathsf{S}^+\mathsf{S} = 1. \qquad \textit{unitarity} \tag{9}$$

(b) [**5 points**] Show that a unitary matrix S can be parameterized by four real parameters α, β, γ, and δ as follows:

$$\mathsf{S} = e^{i\beta}\begin{pmatrix} e^{i\gamma}\cos\alpha & e^{i\delta}\sin\alpha \\ -e^{-i\delta}\sin\alpha & e^{-i\gamma}\cos\alpha \end{pmatrix}. \tag{10}$$

(c) [**5 points**] Show that time-reversal symmetry implies that S must be *symmetric* (**see Hints**):

$$\mathsf{S}^T = \mathsf{S}, \qquad \mathsf{S} = e^{i\beta} \begin{pmatrix} e^{i\gamma}\cos\alpha & i\sin\alpha \\ i\sin\alpha & e^{-i\gamma}\cos\alpha \end{pmatrix}. \qquad \textit{time}-\textit{reversal symmetry} \tag{11}$$

(d) [**5 points**] Show that time-reversal symmetry and conservation of the probability current require the matrix M to have the form:

$$\mathsf{M} = \begin{pmatrix} e^{i\phi}\cosh\theta & e^{i\eta}\sinh\theta \\ e^{-i\eta}\sinh\theta & e^{-i\phi}\cosh\theta \end{pmatrix}. \tag{12}$$

It is not necessary to express ϕ, θ, and η explicitly in terms of α, β, and γ.

(e) [**5 points**] *Adapted from Physics Qualifier, Spring 1967, Problem II-1.*
Consider scattering on a real one-dimensional potential $V(x)$. Is the coefficient of transmission for a particle incident from the left, T_l, the same as transmission coefficient T_r for an identical particle incident from the right with the same energy? (See Problem 4(a) of Homework 3.) Prove your answer. Do *not* assume that the potential $V(x)$ has a certain reflection symmetry (parity).

(f) [**5 points**] Show that if the potential is reflection-symmetric, $V(-x) = V(x)$, then

$$S_{11} = S_{22}, \qquad S_{12} = S_{21}, \qquad \textit{reflection symmetry} \tag{13}$$

and the S-matrix can be parametrized by two real parameters α and β:

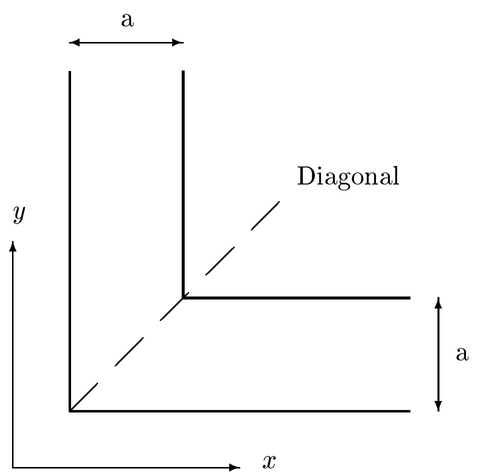
$$\mathsf{S} = e^{i\beta} \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}. \tag{14}$$

Show that the transfer matrix is

$$\mathsf{M} = \begin{pmatrix} e^{i\phi}\cosh\theta & i\sinh\theta \\ -i\sinh\theta & e^{-i\phi}\cosh\theta \end{pmatrix}. \tag{15}$$

6. *Adapted from Physics Qualifier, January 1992, Problem II-3.*

A particle is confined in a two-dimensional guide by infinite straight walls, intersecting at right angles, as shown in the figure by the thick lines.



Far from the junction corner, an energy eigenfunction can be separated into products of transverse and longitudinal modes. Along the horizontal x-channel we have

$$\psi(x,y) = \sum_{n=1}^{\infty} \left[B_n e^{ik_n x} + A_n e^{-ik_n x}\right] \phi_n(y), \qquad x \gg a, \tag{16}$$

while along the vertical y-channel, the same wave function assumes the asymptotic form

$$\psi(x,y) = \sum_{n=1}^{\infty} \left[F_n e^{ik_n y} + G_n e^{-ik_n y}\right] \phi_n(x), \qquad y \gg a, \tag{17}$$

where ϕ_n are the transverse mode wave functions.

Consider the scattering of the particle in the lowest transverse mode. Assume that the longitudinal energy of the a particle is much lower than the transverse excitation energy, so that the scattering can be considered as truly one-dimensional. Define the scattering matrix as

$$\begin{pmatrix} B_1 \\ F_1 \end{pmatrix} = \mathsf{S} \begin{pmatrix} A_1 \\ G_1 \end{pmatrix}. \tag{18}$$

(a) [**3 points**] Identify the symmetries of the system and write the most general form of the scattering matrix compatible with the symmetries. Use the results of Problem 5.

(b) [**5 points**] Suppose that an impenetrable wall of infinite height is added along the diagonal of the guide (the dashed line in the figure). Show that the coefficients A_1 and B_1 are related by a phase shift δ and express δ in terms of the parameters of the scattering matrix found in Problem 6a.

7. *Adapted from Physics Qualifier, fall 1981, Problem II-3.*

To study the electrical resistance of a long wire, one analyzes a simplified model of one-dimensional electrons scattering against a random potential $V(x)$. The potential is characterized by some transfer matrix M, such that the corresponding transmission and reflection coefficients are T and $R = 1 - T$.

(a) [**1 point**] It was suggested that the electrical resistance ρ of such a system is equal to the following ratio:

$$\rho = \frac{R}{T}. \tag{19}$$

It is implied here that ρ is appropriately normalized (divided by e^2/h), so that ρ is a dimensionless number.

What are the values of ρ in the following limiting cases:

i. T=0 (no transmission through the wire),

ii. T=1 (no reflection in the wire)?

Are these limiting values of ρ physically reasonable?

(b) [**3 point**] What is the expression for the ρ (19) in terms of the matrix elements of M?

Suppose, the potential $V(x)$ consists of many randomly-placed identical scatterers, for example randomly placed delta-functions. That seems to be a reasonable model for impurities in metals. Each scatterer is characterized by a given transfer matrix M_1, such that the corresponding transmission and reflection coefficients are T_1 and R_1, and the resistance of the scatterer is $\rho_1 = R_1/T_1$.

(c) [**5 points**] Suppose, the transfer matrix of n consecutive scatterers, M_n, is known. One more scatterer is added at a distance a from the n-th scatterer, so that the total number of scatterers becomes $n+1$.

Using the symmetry restrictions (12) on the form of the transfer matrix ($M_{11} = M_{22}^*$, $M_{12} = M_{21}^*$, and $|M_{11}|^2 - |M_{12}|^2 = 1$) and the multiplication property (Eq. (9) of Homework 3), express the resistance ρ_{n+1} of $n+1$ scatterers in terms of the matrix elements of M_n and M_1.

(d) [**5 points**] Taking into account that the distance a between the scatterers is random, average the obtained expression for ρ_{n+1} over a. Express the averaged ρ_{n+1} in terms of ρ_n and ρ_1.

Solve the recurrent relation between ρ_{n+1} and ρ_n explicitly and find behavior of ρ_n in the limit of small and big n.

You should find that ρ_n increases exponentially at big n. This is a manifestation of the so-called *localization* of electrons in a one-dimensional random potential.

Hints

3c Continue the wave function antisymmetrically to $x > l$. Imagine there is a mirror at $x = l$. The original wave packet goes into the mirror, whereas its mirror reflection comes out of the mirror. (Recall the adventure of Alice in *Through the Looking-Glass* by Lewis Carroll).

5a Use the condition that the probability current must be conserved.

5c Time reversal is complex conjugation.