This is the last homework!

Homework #13 — Phys622 — Fall 1998
Deadline: Friday, December 11, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

Victor Yakovenko, Assistant Professor
Office: Physics 2314
Phone: (301)–405–6151
E-mail: yakovenk@physics.umd.edu

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Addition of Angular Momenta (Chapter 10)

1. [**5 points**] Schwabl's Problem 10.3.

2. [**7 points**] From Problem 4a of Homework 11, we know that the spin-1/2 states up and down along the z axis are transformed upon rotation of the spin quantization axis by the angle θ around the y axis in the following way (see Eqs. (3) and (8) of Homework 11):

$$\begin{aligned} |\uparrow\rangle' &= \cos(\theta/2)|\uparrow\rangle - \sin(\theta/2)|\downarrow\rangle, \\ |\downarrow\rangle' &= \sin(\theta/2)|\uparrow\rangle + \cos(\theta/2)|\downarrow\rangle. \end{aligned} \tag{1}$$

Using Eq. (1), find the transformation law upon the same rotation of the following two-spin-1/2 states:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \qquad |\uparrow\uparrow\rangle, \quad \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |\downarrow\downarrow\rangle. \tag{2}$$

Show that the first state transforms into itself, and the three other states transform between themselves. Compare the 3×3 matrix of the later transformation with the rotation matrix of spin 1, $\hat{R}^{(1)}(-\vartheta\mathbf{e}_y)$, given by Eq. (12) of Problem 5 of Homework 12. Conclude that the first state in Eq. (2) has zero total spin, and the three other states have the total spin one.

Similarly, the state

$$|\underbrace{\uparrow,\uparrow,\ldots,\uparrow}_{2s}\rangle \tag{3}$$

is the eigenstate of the spin s (s is integer or half-integer) with the top value of the z-axis spin projection equal to s. The states with the lower z-axis spin projections can be obtained by applying the lowering operator

$$\hat{S}_- = \sum_{n=1}^{2s} \hat{S}_-^{(n)} \tag{4}$$

to the state (3). In Eq. (4), the lowering operator for the total spin, $\hat{S}_-$, is the sum of the lowering operators for individual spins, $\hat{S}_-^{(n)}$. It is easy to see that all states of the total spin s can be obtained in this manner, and each state is completely symmetric with respect to interchange of any two individual spins. By applying the rotation matrix (1) of spin 1/2 to individual spins, we can find the rotation matrix of the higher spin s. This applies to rotation about any axis, not only the y axis studied in this exercise.

3. [**3 points**] In the hydrogen atom, apart from the Coulomb interaction, there exists another, the so-called *hyperfine* interaction between the magnetic moments of the electron and proton. The Hamiltonian of this interaction is:

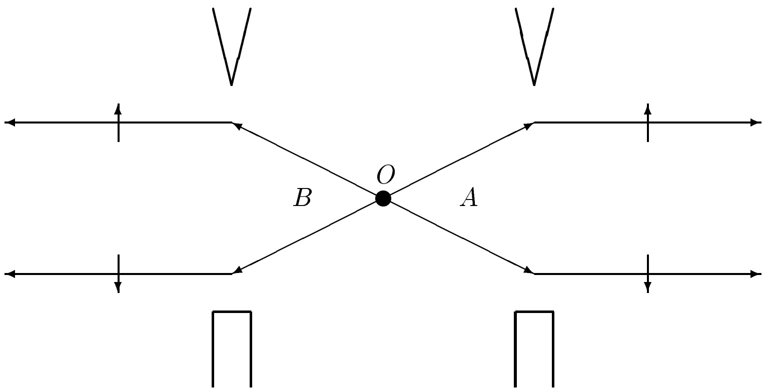
$$\hat{H}_{hf} = A\,\hat{\mathbf{S}}_p \cdot \hat{\mathbf{S}}_e, \qquad A > 0, \tag{5}$$

where $\hat{\mathbf{S}}_p$ and $\hat{\mathbf{S}}_e$ are the spin operators of the proton and electron.

Is the total spin of the electron and proton a good quantum number for Hamiltonian (5)? Find the energy eigenstates and eigenvalues of Hamiltonian (5) and describe their quantum numbers and degeneracies. Describe how the hyperfine interaction (5) splits the ground state energy of the hydrogen atom.

The energy splitting ΔE of the ground state energy of the hydrogen atom caused by the hyperfine interaction (5) is equal to the energy of a photon of the wavelength $\lambda = 21.4$ cm: $\Delta E = hc/\lambda$. Microwave spectroscopy at this frequency is utilized to detect hydrogen in the Universe.

4. A spin-zero particle O decays into two spin-1/2 particle A and B, which fly to the right and to the left, respectively, through the Stern-Gerlach apparatuses, which allow to measure the spin projections of the particles on the z axis, as shown in the figure below.



The wave function of the system after decay of the O particle is

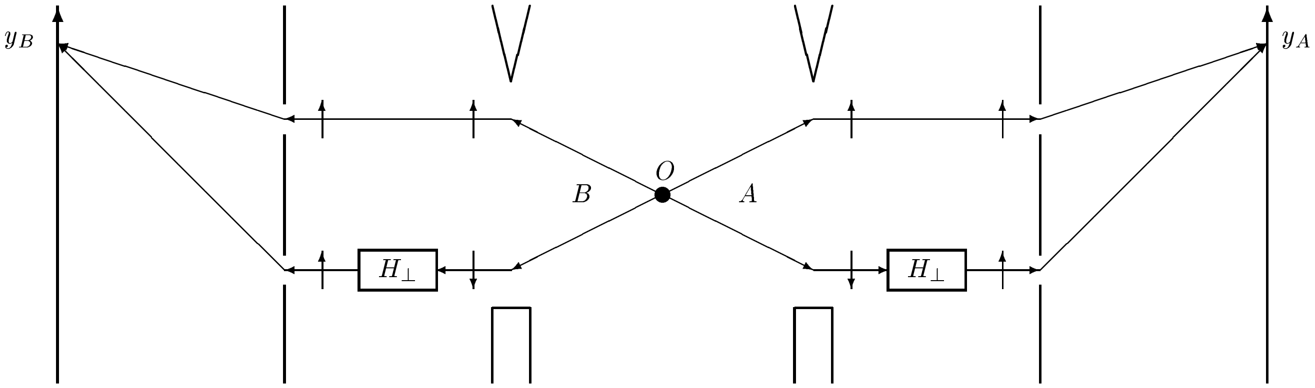
$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \tag{6}$$

where the first arrow refers to the spin projection of particle A and the second to the spin projection of particle B.

(a) [**3 points**] Suppose the spin projection of the A particle is measured, while no measurement is done on the particle B. What are the probabilities to find the particle A in the spin-up and in the spin-down states?

(b) [**3 points**] Suppose the spin projection of the B particle has been measured first and found to be up. Then, the spin projection of the A particle is measured. What are the probabilities to find the particle A in the spin-up and in the spin-down states in this case?

Now, as in Problem 7b of Homework 11, magnetic fields are applied perpendicular to the z axis in the spin-down paths. The strength of the fields is such that the spins of the particles rotate from the down to the up states during the flight time. (In the terminology of Problem 6d of Homework 11, the spin-down particles experience the π pulse.) Then, the beams pass through the spits, and we measure the probability $P(y_A, y_B)$ that the particle A arrives to a point y_A and the particle B to a point y_B within a short period of time ("simultaneously"). The latter condition assures us that both particles originated from the same decay event, not from decay of different O particles.



(c) [**3 points**] Let us consider only those events where particle B arrives to a certain point y_B. For those events, does the probability to find the A particle at an arbitrary point y_A exhibit an interference pattern? (Yes, No)

(d) [**3 points**] Suppose we don't care where the particle B arrives and measure only the probability distribution of the particle A: $P(y_A) = \int dy_B\, P(y_A, y_B)$. Does this function exhibit an interference pattern? (Yes, No)

(e) [**5 points**] Assuming that the distance from the slits to the screens is big compared with both the distance between the slits and the characteristic y's, show that $P(y_A, y_B)$ is a functions of only the relative distance $y_A - y_B$.

The state (6) is a typical example of the so-called entangled state*, a state of several particles whose wave function is not a product of wave functions of individual particles: $\psi(A,B) \neq \psi(A)\psi(B)$.*

5. *There are several algorithms for calculating the Clebsch-Gordan coefficients. One algorithm that involves the lowering operator and orthogonalization is described in Section 10.3 of Schwabl. Another algorithm that involves only applying certain projection operators is described in the book by Sakurai, which, in general, studies the spin and angular momentum in much greater detail. One way or the other, the Clebsch-Gordan coefficients have been calculated and tabulated.*

In this problem, we practice to use the enclosed table of the Clebsch-Gordan coefficients (page 168 from "Introduction to Quantum mecanics" by D. J. Griffiths). Read carefully the instructions how to use the table and answer the following questions using the table.

(a) [**3 points**] Represent the state with the total angular momentum $|j = 0\rangle$ as a superposition of the states with $j_1 = 1$ and $j_2 = 1$.

(b) [**3 points**] One particle of $j_1 = 1$ is in the state $|m_1 = +1\rangle$, and another particle of $j_2 = 1$ is in the state $|m_1 = -1\rangle$. Represent this state as a superposition of the states with certain values of the total angular momentum j.

6. *Adapted from Physics Qualifier, Spring 1974, Problem II-5.*

The Λ^0 particle has spin 1/2 and a definite parity. The Λ^0 particle, which is initially at rest and spin-polarized, decays into a proton p^+, which has spin 1/2, and a pion π^-, a particle of zero spin:

$$\Lambda^0 \to \mathrm{p}^+ + \pi^-. \tag{7}$$

Experimentally, the angular distribution of the pions is found to be proportional to

$$1 + \alpha \cos\theta, \tag{8}$$

where θ is the angle between the direction of motion of the pion and the spin of the Λ^0 particle, and α is a number (say $\alpha = 0.62$).

(a) [**2 points**] Is the parity conserved in this reaction? In other words, is there a violation of the parity symmetry in this reaction or not?

(b) [**3 points**] What are possible values of the *orbital* angular momentum of the pion-proton system in this reaction? (Take into account conservation of the total angular momentum and don't forget to count the spin of proton toward the total angular momentum.)

Using the table of Clebsch-Gordan coefficients, write the wave function of the pion-proton system as a Dirac ket vector. The wave function must reflect the orbital state of the system and the spin state of the proton. Denote the unknown coefficients in front of the terms with the orbital momenta $l = 0$ and $l = 1$ by a and b. Other coefficients should be fixed by the Clebsch-Gordan table. Overall normalization of the wave function is not required.

(c) [**3 points**] Using the wave function found in Problem 6b, find the angular distributions of pions in the following two cases:

i. The spin of proton has been measured to be parallel to the spin of Λ^0;

ii. The spin of proton has been measured to be antiparallel to the spin of Λ^0.

Hint: You may use the spherical wave functions (Eq. (5.34) of Schwabl)

$$Y_{00} = \sqrt{\frac{1}{4\pi}}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

assuming that the angles θ and ϕ characterize the direction of pion's *momentum* $\mathbf{p}$ instead of the coordinate $\mathbf{r}$.

(d) [**3 points**] Equate the sum of the two angular distributions found in Problem 6c to Eq. (8) and find the ratio of a and b in terms of α. Determine the ratio of probabilities to find the pion in the states with $l = 0$ and $l = 1$.