Victor Yakovenko, Assistant Professor
Office: Physics 2314
Phone: (301)–405–6151
E-mail: yakovenk@physics.umd.edu

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Spin (Chapter 9)

1. *Adapted from Qualifier, Spring 1985, II-5, September 1989 and Fall 1983, II-2.*

In this problem, we explore theoretical foundations of magnetic spin resonance, which is widely used from solid state physics and material science to biology and medicine to study properties of matter.

Consider electron (a spin-1/2 particle) in a magnetic field $\mathbf{B}$. Time evolution of the spin degrees of freedom is described by the Pauli equation (9.41):

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \mu_B(\mathbf{B}\cdot\hat{\boldsymbol{\sigma}})\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}. \tag{1}$$

where ψ_+ and ψ_- are the spinor components of the electron wave function corresponding to the spin up and down states relative to the z axis. In Problem 6 of Homework 11, we studied the case where the magnetic field is constant in time: $\mathbf{B} = B_0\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in the z direction, and $B_0 > 0$. Now, let us add a time-dependent magnetic field $\mathbf{B}_\perp(t)$ that is perpendicular to the z axis and rotates with the frequency ω, so that the total magnetic field becomes

$$\mathbf{B}(t) = B_0\mathbf{e}_z + B_1\mathbf{e}_\perp(t), \tag{2}$$

where $\mathbf{e}_\perp(t)$ is the rotating unit vector:

$$\mathbf{e}_\perp(t) = \mathbf{e}_x\cos(\omega t) + \mathbf{e}_y\sin(\omega t). \tag{3}$$

(a) [**5 points**] Find the time evolution equation for the spinor wave function $\tilde{\psi}(t)$ in the reference frame rotating about the z axis with the frequency ω. That is, represent the spinor wave function $\psi(t)$ in the laboratory reference frame in the form:

$$\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = \hat{R}(-\omega t\mathbf{e}_z)\begin{pmatrix}\tilde{\psi}_+(t)\\ \tilde{\psi}_-(t)\end{pmatrix}, \tag{4}$$

where $\hat{R}(-\omega t\mathbf{e}_z)$ is the operator of rotation about the z axis given by Eq. (2) of Homework 11; substitute Eq. (4) into Eq. (1); and find the equation for $\tilde{\psi}(t)$. **(see Hints)**

Identify and sketch the effective magnetic field in the rotating reference frame and show that it is stationary (time-independent). What is the difference between the z components of the magnetic field in the laboratory and the rotating reference frames?

The appearance of an additional magnetic field in the rotating reference frame constitutes the Larmor theorem.

(b) [**5 points**] Find the time evolution operator $\hat{\tilde{U}}(t)$ for $\tilde{\psi}(t)$ in the rotating reference frame and the time evolution operator $\hat{U}(t)$ for $\psi(t)$ in the laboratory reference frame.

(c) [**5 points**] Suppose at $t = 0$ the electron has spin up: $\psi(t=0) = \binom{1}{0}$. Find the spinor wave function of the electron at a time t in the laboratory reference frame and the probability to flip the spin down (*the Rabi formula*).

(d) [**3 points**] What is special about the resonant case where the rotation frequency of the transverse magnetic field $\mathbf{B}_\perp(t)$ is equal to the precession frequency in the longitudinal field B_0: $\omega = \omega_{B_0}$, where ω_{B_0} was found in Problem 6a of Homework 11?

2. [**3 points each part**] Read the enclosed pages 299 and 300 from R. Shankar "Principles of Quantum Mechanics" (second edition) about parity violation in weak interactions and do the exercises from these pages:

(a) Exercise 11.4.1.

(b) Exercise 11.4.3.

(c) Exercise 11.4.4.

3. [**7 points**] Suppose an extraterrestrial intelligence (therefrom referred as "they" or "them") has been discovered somewhere in the Universe. The humans of the Earth (therefrom referred as "we" or "us") have established contact with them and want to send them a picture of a human. This can be done by transmitting an array of coordinates (x, y, z) that describe our shape. Particularly, we want to explain them that we have hearts on our left sides.

In order to use our data, they have to choose a Cartesian reference frame. The x axis can be selected arbitrarily; the y axis can be selected along an arbitrary direction perpendicular to the x direction; the z axis should be selected along the line perpendicular both the x and y axes. However, the direction of the z axis can be selected in two opposite ways, one making the reference frame right-handed and another left-handed. If they choose the left-handed orientation, the image reconstructed from our data will be a mirror reflection of us, with the heart on the right side.

Describe how we could instruct them to unambiguously select the right-handed reference frame by using the fundamental physical processes discussed in Problem 2. (You cannot refer to clocks, screws, or right hands to explain them right-handedness, because they have never seen these objects.)

Assume that they are made of the same matter that we are. Would your instructions lead to identification of the correct reference frame if they are made of antimatter (positrons instead of electrons, etc.)? Antiparticles have the same properties as usual particles except their electric charges are opposite.

4. *Adapted from Qualifier, Fall 1994, II-4.*

Let us discuss transformation of spin-1/2 wave functions upon the time reversal operation.

(a) [**5 points**] Upon the time reversal, not only the wave function should be complex-conjugated, but also the angular momentum should change sign. Spin up should become spin down and vise versa, thus the components of a spinor need to be interchanged. That must be true for any choice of the spin quantization axis.
Show that the following is the operator of the time reversal for a spin-1/2 wave function

$$\hat{T} = -i\hat{\sigma}_y \hat{K} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{K}, \tag{5}$$

where $\hat{K}$ is the operator of complex conjugation. That is, demonstrate that applying $\hat{T}$ to $|\mathbf{e},+\rangle$ (see Problem 4 of Homework 11) we get $|\mathbf{e},-\rangle$ and vice versa for any direction $\mathbf{e}$.

The operators $\hat{K}$ and $\hat{T}$ are the so-called *antilinear* operators, which have the following property:

$$\hat{K}(\alpha|a\rangle + \beta|b\rangle) = \alpha^* \hat{K}|a\rangle + \beta^* \hat{K}|b\rangle, \tag{6}$$

different from the similar property of a linear operator $\hat{L}$:

$$\hat{L}(\alpha|a\rangle + \beta|b\rangle) = \alpha \hat{L}|a\rangle + \beta \hat{L}|b\rangle, \tag{7}$$

(b) [**3 points**] Using Eq. (5), show that

$$\hat{T}^2 = -1. \tag{8}$$

(c) [**5 points**] Suppose that a Hamiltonian $\hat{H}$ of a spin-1/2 system has the time-reversal symmetry, thus $\hat{H}$ and $\hat{T}$ commute:

$$\hat{H}\hat{T} = \hat{T}\hat{H}. \tag{9}$$

Show that each energy eigenstate is at least double-degenerate. **(see Hints)**

This is the so-called Kramers degeneracy. *Since any higher spin can be represented as a composition of spins 1/2, Eq. (8) and the statement 4c are valid for any half-integer spin. The textbooks by Sakurai and Merzbacher discuss the time-reversal symmetry for spins in more detail.*

5. Let us discuss the properties of the angular momentum operator $\hat{\mathbf{J}}$ with the eigenvalue $j = 1$. That is, let us study the properties of the angular momentum operator $\hat{\mathbf{J}}^{(1)}$ acting in the $(2j+1) = 3$-dimensional subspace of the the wave functions that are the eigenfunctions of $\hat{\mathbf{J}}^2$ with the eigenvalue $\hbar^2 j(j+1) = 2\hbar^2$. We use the letter $\hat{\mathbf{J}}$ here in order to emphasize that the results apply to both the orbital angular momentum $\hat{\mathbf{L}}$ and to the internal angular momentum (spin) $\hat{\mathbf{S}}$.

(a) [**3 points**] Schwabl's Problem 8.4. Do only the case $l = 1$. Write the operators $\hat{J}_x^{(1)}$, $\hat{J}_y^{(1)}$, and $\hat{J}_z^{(1)}$ as 3×3 matrices in the representation of the eigenfunctions of $\hat{J}_z^{(1)}$.
Check for yourself, but don't write in the homework, that in the case $j = 1/2$ the operators $\hat{J}_x^{(1/2)}$, $\hat{J}_y^{(1/2)}$, and $\hat{J}_z^{(1/2)}$ are the 2×2 Pauli matrices times $\hbar/2$.

(b) [**3 points**] Show that

$$\left(\frac{\hat{J}_x^{(1)}}{\hbar}\right)^3 = \frac{\hat{J}_x^{(1)}}{\hbar}, \qquad \left(\frac{\hat{J}_y^{(1)}}{\hbar}\right)^3 = \frac{\hat{J}_y^{(1)}}{\hbar}. \tag{10}$$

(c) [**5 points**] For $j = 1$, the operator of rotation by the angle φ about an axis of rotation along a unit vector $\mathbf{n}$ is given by the standard expression

$$\hat{R}^{(1)}(\varphi\mathbf{n}) = e^{i\varphi\mathbf{n}\cdot\hat{\mathbf{J}}^{(1)}/\hbar}. \tag{11}$$

Using Eq. (10), show that

$$\begin{aligned}\hat{R}^{(1)}(\vartheta\mathbf{e}_x) &= \hat{I}^{(1)} + i\sin\vartheta\,\frac{\hat{J}_x^{(1)}}{\hbar} + (\cos\vartheta - 1)\left(\frac{\hat{J}_x^{(1)}}{\hbar}\right)^2, \\ \hat{R}^{(1)}(\vartheta\mathbf{e}_y) &= \hat{I}^{(1)} + i\sin\vartheta\,\frac{\hat{J}_y^{(1)}}{\hbar} + (\cos\vartheta - 1)\left(\frac{\hat{J}_y^{(1)}}{\hbar}\right)^2.\end{aligned} \tag{12}$$

(d) [**3 points**] Using Eq. (12), write the rotation matrices $\hat{R}^{(1)}(\vartheta\mathbf{e}_x)$ and $\hat{R}^{(1)}(\vartheta\mathbf{e}_y)$ explicitly as 3×3 matrices in the representation of the eigenfunctions of $\hat{J}_z^{(1)}$.

6. *Adapted from Physics Qualifier, January 1997, Problem II-4.*

(a) [**3 points**] For a spin-1 particle, represent the eigenstate of $\hat{S}_z$ with the eigenvalue $+\hbar$ as a superposition of the eigenstates of $\hat{S}_{z'}$ with the eigenvalues $+\hbar$. 0, and $-\hbar$ relative the axis of quantization z' tilted to the angle ϑ relative to the z axis and find the coefficients of superposition. Use the result of Problem 5d.

(b) [**3 points**] A beam of spin-1 particles in the eigenstate of $\hat{S}_z$ with the eigenvalue $+\hbar$ passes through a Stern-Gerlach apparatus tilted to the angle ϑ relative to the z axis. The beam splits into three beams corresponding to the eigenstates of $\hat{S}_{z'}$ with the eigenvalues $+\hbar$. 0, and $-\hbar$ relative the axis of quantization z' tilted to the angle ϑ relative to the z axis.
Determine in what proportion the initial beam splits between the three subsequent beams.

Hints

1a Don't forget to differentiate $\hat{R}(-\omega t\mathbf{e}_z)$ over time. Use Eq. (6) of Homework 11 to find the transformed Hamiltonian.

4c Apply the operator $\hat{T}$ to an energy eigenstate $|\psi\rangle$ and show that the resulting state is an energy eigenstate of the same eigenvalue, but it cannot be written as $e^{i\varphi}|\psi\rangle$, thus does not coincide with $|\psi\rangle$. Use Eqs. (6) and (8).