Homework #11 — Phys622 — Fall 1998
Deadline: Friday, November 20, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Spin (Chapter 9)

1. [**3 points**] Schwabl's Problem 9.1.

2. [**5 points**] Schwabl's Problem 9.2.

3. [**5 points each part**] Eq. (5.3) gives the operator of infinitesimal rotations. The operator of a finite rotation by the angle φ about an axis of rotation along a unit vector $\mathbf{n}$ is

$$\hat{R}(\varphi\mathbf{n}) = e^{i\varphi\mathbf{n}\cdot\hat{\mathbf{J}}/\hbar}, \tag{1}$$

where $\hat{\mathbf{J}}$ is the angular momentum operator, the generator of rotations (see Table 8.1). The angular momentum operator $\hat{\mathbf{J}}$ in Eq. (1) may be the operator $\hat{\mathbf{L}}$ of orbital angular momentum (5.1) or the operator $\hat{\mathbf{S}}$ of internal angular momentum (spin). The eigenvalues l of the former operator are integer, whereas the eigenvalues s of the latter may be also half-integer.

In this and the following problems, we will examine the rotation operator (1) in detail in the case of $s = 1/2$, where it can be written in terms of the Pauli matrices as follows:

$$\hat{R}(\varphi\mathbf{n}) = e^{i\varphi\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}/2}. \tag{2}$$

The operator (2) is a 2×2 matrix that transforms the two-component spinor wave functions upon rotation.

(a) Show that **(see Hints)**

$$\hat{R}(\varphi\mathbf{n}) = \cos(\varphi/2) + i\,(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\sin(\varphi/2). \tag{3}$$

(b) [**2 points**] What is $\hat{R}(\varphi = 2\pi)$? $\hat{R}(\varphi = 4\pi)$?

(c) Using Eq. (3) and Eq. (9.18b), show that

$$\hat{R}(\varphi\mathbf{n})\,\hat{\boldsymbol{\sigma}}\,\hat{R}^{+}(\varphi\mathbf{n}) = \mathbf{n}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}) - \mathbf{n}\times[\mathbf{n}\times\hat{\boldsymbol{\sigma}}]\cos\varphi + [\mathbf{n}\times\hat{\boldsymbol{\sigma}}]\sin\varphi. \tag{4}$$

(d) Show that, upon rotation by the angle φ about an axis of rotation along a unit vector $\mathbf{n}$, a vector $\mathbf{a}$ transforms into the vector

$$\mathbf{a}' = \mathbf{n}(\mathbf{n}\cdot\mathbf{a}) - \mathbf{n}\times[\mathbf{n}\times\mathbf{a}]\cos\varphi + [\mathbf{n}\times\mathbf{a}]\sin\varphi. \tag{5}$$

(This is a purely geometrical question. To answer it, use geometry, not quantum mechanics.) Compare Eqs. (4) and (5) and conclude that the vector operator $\hat{\boldsymbol{\sigma}}$ transforms upon rotation in the same manner as any other vector.

(e) Using Eqs. (3) and (5), show that

$$\hat{R}(-\varphi\mathbf{n})\,(\mathbf{a}\cdot\hat{\boldsymbol{\sigma}})\,\hat{R}^{+}(-\varphi\mathbf{n}) = \mathbf{a}'\cdot\hat{\boldsymbol{\sigma}}. \tag{6}$$

It is assumed here that the vector $\mathbf{a}$ is not an operator and commutes with the spin operators and the Pauli matrices.

4. [**5 points each part**]

(a) Let us discuss the eigenfunctions of the projection $\mathbf{e}\cdot\hat{\mathbf{S}}$ of the spin operator onto the unit vector

$$\mathbf{e} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta), \tag{7}$$

whose orientation is characterized by the Euler angles ϑ and φ.
Using Eq. (6), argue that the wave functions

$$|\mathbf{e},+\rangle = \hat{R}(-\varphi\mathbf{e}_z)\,\hat{R}(-\vartheta\mathbf{e}_y)\begin{pmatrix}1\\0\end{pmatrix}, \qquad |\mathbf{e},-\rangle = \hat{R}(-\varphi\mathbf{e}_z)\,\hat{R}(-\vartheta\mathbf{e}_y)\begin{pmatrix}0\\1\end{pmatrix}, \tag{8}$$

where $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors along the y and z axis, are the eigenfunctions of $\mathbf{e}\cdot\hat{\mathbf{S}}$ with the eigenvalues $\pm\hbar/2$:

$$\mathbf{e}\cdot\hat{\mathbf{S}}\,|\mathbf{e},\pm\rangle = \pm\frac{\hbar}{2}\,|\mathbf{e},\pm\rangle. \tag{9}$$

(b) Verify the result of Problem 4a by explicitly calculating $|\mathbf{e},\pm\rangle$ (perform the multiplications in Eq. (8) explicitly using Eq. (3)) and $\mathbf{e}\cdot\hat{\mathbf{S}}$ (multiply Eq. (7) by $\hat{\boldsymbol{\sigma}}$) and applying $\mathbf{e}\cdot\hat{\mathbf{S}}$ to $|\mathbf{e},\pm\rangle$.

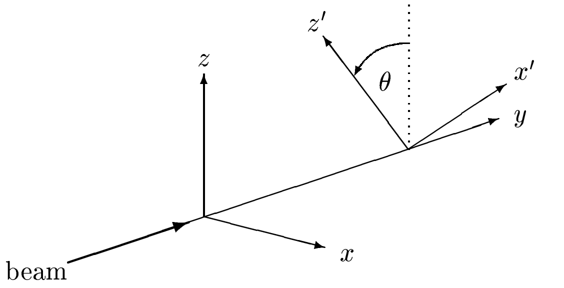
(c) Show that for any spinor $\psi = \binom{c_1}{c_2}$ it is possible to find a direction $\mathbf{e}$ such that ψ is the eigenfunction of $\mathbf{e}\cdot\hat{\mathbf{S}}$ with the eigenvalues $+\hbar/2$. In other words, the spin of a spin-1/2 particle is always completely polarized along a certain direction $\mathbf{e}$. *This statement is not true for the angular momentum higher than 1/2.*

Thus, a normalized wave functions of a spin-1/2 particle can be characterized either by the spinor components c_1 and c_2 (three real parameters), or by the Euler angles ϑ and φ of the polarization vector $\mathbf{e}$ and the overall phase of the wave functions (also three real parameters). (You do not need to find an explicit relation between these two sets of parameters.)

5. [**3 points each part**] *Physics Qualifier, January 1997, Problem II-4.*

(a) A beam of spin-1/2 particles moving along the y axis passes through two collinear Stern-Gerlach apparatuses, both with the lower beams blocked (the particles in the lower beam are absorbed). The first apparatus has its magnetic field along the z axis and the second has its magnetic field along the axis z' tilted to the angle ϑ relative to the z axis in the (x,z) plane.

What fraction of the particles leaving the first apparatus will exit the second one?



(b) If a third Stern-Gerlach apparatus that transmits only spin up along the z axis is introduced, what fraction of particles leaving the first apparatus will exit the third?

(c) If the middle filter transmits both spins up and down (no blocking), what fraction of particles leaving the first apparatus will exit the third?

6. [**3 points each part**] Consider electron (a spin-1/2 particle) in a constant magnetic field $\mathbf{B}$. In this problem, we are interested in the time evolution of the spin degrees of freedom, which is described by the Pauli equation (9.41):

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_+\\\psi_-\end{pmatrix} = \mu_B(\hat{\boldsymbol{\sigma}}\cdot\mathbf{B})\begin{pmatrix}\psi_+\\\psi_-\end{pmatrix}. \tag{10}$$

(a) Show that Eq. (10) generates the time evolution of the spinor wave function,

$$\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = \hat{U}(t)\begin{pmatrix}\psi_+(0)\\ \psi_-(0)\end{pmatrix}, \tag{11}$$

corresponding to rotation of the spinor wave function about the direction of the magnetic field with a frequency ω_B. (Express the time evolution operator $\hat{U}(t)$ in terms of the rotation operator $\hat{R}$ (2)). Find ω_B.

(b) Show that the Ehrenfest equations (see Sec. 2.6) for the average spin $\langle\mathbf{S}\rangle$ of electron are

$$\frac{d\langle\hat{\mathbf{S}}\rangle}{dt} = \frac{2\mu_B}{\hbar}\mathbf{B}\times\langle\hat{\mathbf{S}}\rangle = \langle\hat{\boldsymbol{\mu}}\rangle\times\mathbf{B}, \tag{12}$$

where $\hat{\boldsymbol{\mu}} = (ge/2mc)\hat{\mathbf{S}}$ is the magnetic moment of the electron. Compare the evolution produced by Eq. (12) with the evolution described by Eq. (11).

(c) At the initial moment of time $t = 0$, the electron wave function is equal to

$$\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{13}$$

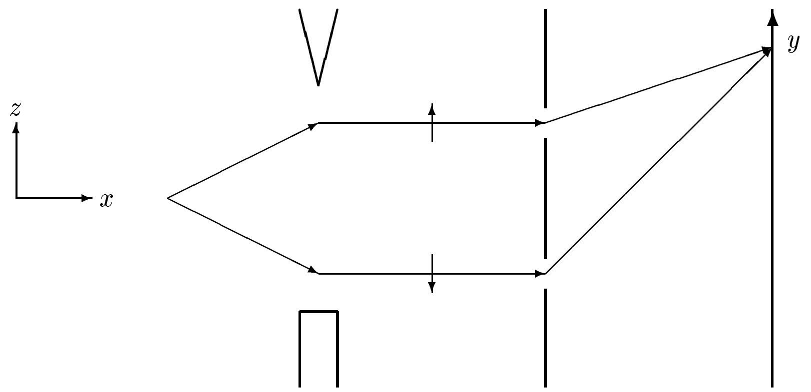
Find the wave function at a time t. Find the spin polarization of electron at the initial moment and at the time t.

(d) Suppose a constant magnetic field $\mathbf{B}$ is applied for a time t in the direction perpendicular to the initial polarization of the electron. What is the time t_π required to flip the spin polarization to the opposite direction (the so-called π-pulse)? What is the time $t_{2\pi}$ such that the spin polarization rotates by the angle 2π and returns to the original orientation (the 2π-pulse)?

The two quantum states of a spin 1/2 with respect to the z axis, $|\uparrow\rangle$ and $|\downarrow\rangle$, can be associated with one bit, 1 and 0 (the so-called qubit*). The π pulse of a magnetic field performs the logical NOT operation: $1 \to 0$ and $0 \to 1$. Using a pair of spins, it is possible to realize logical operations where binary output depends on two binary inputs. This could be a basis for building quantum computers.*

7. When spinless particles pass through two slits, they exhibit an interference pattern (see Fig. 2.1c). Let us discuss how this picture is modified when the particles have spin 1/2.

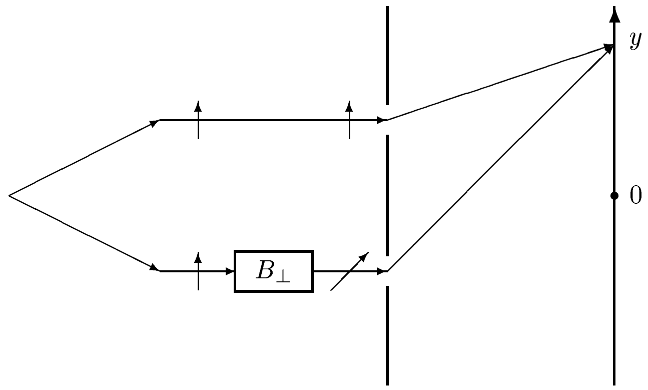
(a) [**3 points**] A beam of spin-1/2 particles with some generic polarization (not along the z axis) passes through a Stern-Gerlach apparatus (shown in the Figure below as the wedge and the rectangle). Then, the spin-up and spin-down beams pass through separate slits, and the probability $P(y)$ to register the particles is measured as a function of the coordinate y.



Does $P(y)$ exhibit an interference pattern for the spin-1/2 particles in the setup described above? (Yes, No)

Describe qualitatively the spin polarization of the particle as a function of the coordinate y.

(b) A beam of spin-1/2 particles polarized up along the z axis is split into two parts. One part passes through a region with a magnetic field $B_\perp$ perpendicular to the z axis, as shown in the Figure below by the rectangle with $B_\perp$.



During the flight through the region with the magnetic field, the polarization of the particles rotates by an angle θ, which depends on the strength of the magnetic field. By varying magnetic field, we can change the rotation angle θ.

i. [**3 points**] Describe qualitatively how the amplitude of the interference pattern depends on θ.

ii. [**3 points**] In zero magnetic field ($\theta = 0$), the interference pattern has the maximum at the point $y = 0$ between the two slits.
When $\theta = 2\pi$, does the interference pattern have a maximum at the point $y = 0$? What is the answer when $\theta = 4\pi$? Compare your result with the answer to Problem 3b.

Hints

3a Expand the exponent in Eq. (2) in the Taylor series and use Eq. (9.18b).