Homework #10 — Phys622 — Fall 1998
Deadline: Friday, November 13, 1998.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Momentum Representation (Chapter 8)

1. **[3 points]** Schwabl's Problem 8.3.

2. Suppose we know the wave function $|\psi(t_1)\rangle$ at the time t_1. The wave function at another time t_2, $|\psi(t_2)\rangle$, is related to the original wave function by the time evolution operator $\hat{U}(t_2,t_1)$ (see Problem 6 of Homework 1):
$$|\psi(t_2)\rangle = \hat{U}(t_2,t_1)|\psi(t_1)\rangle. \tag{1}$$
The time evolution operator can be expanded over a complete set of the energy eigenfunctions $|n\rangle$ with the energy eigenvalues E_n:
$$\hat{U}(t_2,t_1) = e^{-i\hat{H}(t_2-t_1)/\hbar} = \mathop{\mathbf{S}}_n e^{-iE_n(t_2-t_1)/\hbar}\,|n\rangle\langle n|. \tag{2}$$
   (a) **[5 points]** For the homogeneous electric field studied in Problem 1, find the time evolution operator explicitly in the momentum representation **(see Hints)**:
$$\hat{U}(p_2,t_2;p_1,t_1) = \mathop{\mathbf{S}}_n e^{-iE_n(t_2-t_1)/\hbar}\,\langle p_2|n\rangle\langle n|p_1\rangle. \tag{3}$$
   Does the result make sense physically?
   (b) **[2 points]** Write how to obtain the time evolution operator in the coordinate representation
$$\hat{U}(x_2,t_2;x_1,t_1) = \mathop{\mathbf{S}}_n e^{-iE_n(t_2-t_1)/\hbar}\,\langle x_2|n\rangle\langle n|x_1\rangle \tag{4}$$
   from the time evolution operator in the momentum representation found in the previous part 2a. To save time, you are not required to take all of the integrals explicitly, even though they are doable in this case.

3. **[5 points]** Find the energy and the wave function $\varphi(p)$ of a bound state of the one-dimensional delta-function potential:
$$V(x) = -\lambda\,\delta(x), \qquad \lambda > 0, \tag{5}$$
using the Schrödinger equation in the momentum representation **(see Hints)**. Compare the result with your solution in the coordinate representation.

4. *Adapted from Physics Qualifier, August 1982, Problem II-1.*
   (a) **[5 points]** An electron, confined to two-dimensional (2D) motion at the surface of a crystal, is subject to an attractive delta-function potential $V(\mathbf{r}) = -\lambda\delta^{(2)}(\mathbf{r})$, where $\lambda > 0$ and $\mathbf{r} = (x, y)$. Because of band structure considerations, electron's momentum is subject to the condition $|\mathbf{p}| < p_0$, that is $\varphi(p) = 0$ for $|\mathbf{p}| > p_0$.
   Using the Schrödinger equation in the momentum representation, find the wave function $\varphi(\mathbf{p})$ and the energy of a bound state of the electron.
   (b) **[5 points]** Consider a *shallow* cylindrically-symmetrical potential well $V(\mathbf{r})$ in 2D. The term "shallow" was defined in Problem 2c of Homework 3. It means that the characteristic depth V_0 and width a of the potential $V(\mathbf{r})$ obey the following inequality:[1)]
$$\hbar^2/ma^2 \gg V_0, \tag{6}$$

[1)] The presence of the parameters V_0 and a does *not* imply that the shape of the potential well is rectangular. The word "characteristic" indicates that these are the parameters that characterize the overall size of the potential $V(\mathbf{r})$, which may have an arbitrary shape.

where m is the mass of the particle moving in this potential.

Using your solution of Problem 4a, find (approximately) the ground bound state energy of the electron subject to this potential. Assume that the wave function is cylindrically symmetrical. Unlike in Problem 4a, there is no restriction on the electron's momentum in this problem. **(see Hints)**

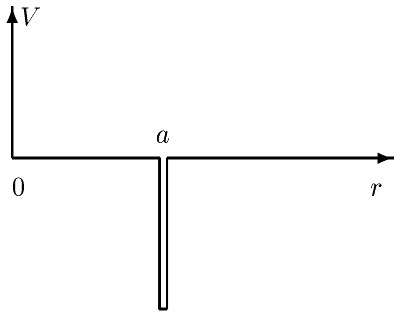
(c) [**3 points**] Taking the result of Problem 4b into account, argue that any attractive potential well has at least one bound state in 2D (as well as in 1D).

5. *Adapted from Physics Qualifier, January 1982, Problem II-1.*

Consider a quantum particle of mass m moving in a three-dimensional spherically-symmetrical attractive potential that vanishes everywhere except at the radius $r = a$:

$$U(r) = -\lambda\delta(r-a), \qquad \lambda > 0, \tag{7}$$

where δ is the Dirac δ-function.



(a) [**5 points**] Using the Schrödinger equation in the momentum representation, find a transcendental equation for the energy of a spherically-symmetrical bound state.

(b) [**3 points**] Show that the equation found in Problem 5a does not have solutions for $\lambda < \lambda_{\rm min}$ and find $\lambda_{\rm min}$. Conclude that a sufficiently weak potential well in three dimensions does not have a bound state, unlike in one-dimensional case.

6. [**5 points**] Let us modify the previous Problem 5 to the extent that the wave function $\varphi(\mathbf{p})$ of the particle is subject to the following boundary condition in the momentum space:

$$\varphi(\mathbf{p}) = 0, \qquad \text{when } |\mathbf{p}| \le p_0, \qquad p_0 > 0. \tag{8}$$

That is, the particle cannot occupy the states with the momenta smaller than p_0. In this case, the bound state would be a state that has an energy lower that the energy level of the vacuum, $E_0 = p_0^2/2m$.

Show that, if condition (8) is imposed, the potential well (7) has a bound state, no matter how small λ is. In the case where the attractive potential is weak:

$$\lambda \ll \hbar p_0/m, \tag{9}$$

find (approximately) the energy of the bound state and compare it with the vacuum energy level E_0.

[*In this problem, p_0 mimics the effect of the Fermi surface in a metal. Inside the Fermi surface ($p < p_0$) all quantum states are occupied by other electrons, so the wave function of a probe electron must obey condition (8). Introduction of the Fermi surface allows an arbitrarily weak attractive potential to produce a bound state, which is not the case in three dimensions without the Fermi surface. More generally, an effective attractive potential (mediated by phonons) acts between different electrons and binds them in pairs. This phenomenon, which is an important ingredient of the theory of superconductivity, was discovered by Leon Cooper, who is C in BCS. Full solution of the Cooper problem can be found in Chapter 8 of the book by Baym.*]

7. [**3 points**] Suppose a particle is in the eigenstate $|lm\rangle$ of the angular momentum operators $\hat{\mathbf{L}}^2$ and $\hat{L}_z$. Thus, the angular dependence of the wave function in the coordinate representation is given by the spherical harmonics: $\langle \mathbf{r}|lm\rangle = \psi_{lm}(\mathbf{r}) = \psi_{lm}(r,\theta,\varphi) = Y_{lm}(\theta,\varphi)R(r)$, where $R(r)$ is some radial function.

What is the angular dependence of the corresponding wave function $\langle \mathbf{p}|lm\rangle = \tilde{\psi}_{lm}(\mathbf{p})$ in the momentum representation? In other words, how does the function $\tilde{\psi}_{lm}(\mathbf{p}) = \tilde{\psi}_{lm}(p,\tilde{\theta},\tilde{\varphi})$ depend on the Euler angles $\tilde{\theta}$ and $\tilde{\varphi}$ of the vector $\mathbf{p}$? **(see Hints)**

Hints

2a Normalize the energy eigenfunctions $\varphi_E(p)$ so that $\langle \varphi_E|\varphi_{E'}\rangle = \delta(E-E')$, where the energy E is not restricted to be positive. Then, the summation over n in Eq. (2) can be replaced by integration over E.

3 Multiply the Schrödinger equation in the coordinate representation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \lambda\delta(x)\right]\psi(x) = E\psi(x) \tag{10}$$

by $\exp(-ipx/\hbar)$ and integrate over x to obtain the Schrödinger equation in the momentum representation

$$\frac{p^2}{2m}\varphi(p) - \lambda\psi(0) = E\varphi(p). \tag{11}$$

Find $\varphi_E(p)$ from Eq. (11) and use the self-consistency relation

$$\psi(0) = \int \frac{dp}{2\pi\hbar}\varphi(p) \tag{12}$$

to determine the energy E of the bound state.

4b Similarly to the one-dimensional case, a *shallow* well in two dimensions can be treated as the delta-function potential. The inverse width of the potential well, $\hbar/a$, plays the role of the momentum cutoff p_0 in this case.

7 The angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ has the same form in the coordinate: $\hat{\mathbf{L}} = -i\hbar\,\mathbf{r}\times\partial/\partial\mathbf{r}$, and the momentum: $\hat{\mathbf{L}} = -i\hbar\,\mathbf{p}\times\partial/\partial\mathbf{p}$, representations.