Homework #1 — Phys622 — Fall 1998
Deadline: Wednesday, September 9, 1998.
Turn in homework in the class or put
in the box on the door of Phys 2314.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Wave Functions, Operators, and the Uncertainty Relation (Chapters 1, 2, 4)

1. [**1 point each part**] Schwabl's Problem 2.4.

2. [**2 points each part**] Schwabl's Problem 2.5. Show also that under the assumption of part (a) the following identity holds:
$$[e^B, A] = e^B[B, A]. \tag{1}$$

3. [**4 points**] Schwabl's Problem 2.8. Use the results of Problem 1.

4. **(a)** [**2 points**] Show that, with the wave function written as $\psi(\mathbf{r}) = |\psi(\mathbf{r})|e^{i\alpha(\mathbf{r})}$, the probability current density (Schwabl's Eq. (2.59)) is
$$\mathbf{j}(\mathbf{r}) = \frac{\hbar}{m}|\psi(\mathbf{r})|^2\boldsymbol{\nabla}\alpha(\mathbf{r}). \tag{2}$$

   (b) [**2 points**] Show that, if the wave function $\psi(\mathbf{r})$ is proportional to a real function $\psi_r(\mathbf{r})$: $\psi(\mathbf{r}) = c\psi_r(\mathbf{r})$, where c is a constant complex number, then the probability current density vanishes everywhere: $\mathbf{j}(\mathbf{r}) = 0$, and the expectation value of momentum vanishes: $\langle\hat{\mathbf{p}}\rangle = 0$.

   (c) [**2 points**] A real-space wave function $\psi_1(\mathbf{r})$ corresponds to the wave function $\varphi_1(\mathbf{p})$ in the momentum space. Show that the real-space wave function $\psi_2(\mathbf{r}) = \exp(i\mathbf{p}_0\cdot\mathbf{r}/\hbar)\psi_1(\mathbf{r})$ corresponds to the displaced momentum-space wave function $\varphi_2(\mathbf{p}) = \varphi_1(\mathbf{p}-\mathbf{p}_0)$, and the expectation values of momentum in the two states are related by the formula $\langle\hat{\mathbf{p}}\rangle_2 = \mathbf{p}_0 + \langle\hat{\mathbf{p}}\rangle_1$. Sketch $\varphi_1(\mathbf{p})$ and $\varphi_2(\mathbf{p})$ for an one-dimensional Gaussian packet assuming that $p_0 > 0$.

5. [**3 points total**] Schwabl's Problem 2.7.

6. Suppose we know the wave function $\psi(x,t_1)$ at the time t_1. The wave function at another time t_2, $\psi(x,t_2)$, is related to the original wave function by a linear operator U:
$$\psi(x,t_2) = \int dx'\, U(x,t_2,x',t_1)\psi(x',t_1). \tag{3}$$
The operator $U(x,t_2,x',t_1)$ is called the *time evolution operator*.

   (a) [**2 points**] Show that
$$U(x,t_2,x',t_1) = \mathop{\mathbf{S}}_{n} e^{-iE_n(t_2-t_1)/\hbar}\varphi_n(x)\varphi_n^*(x'), \tag{4}$$
where the generalized sum $\mathop{\mathbf{S}}_{n}$ (see Eq. (2.101) of Schwabl) is taken over a complete set of the energy eigenfunctions $\varphi_n(x)$ with the energy eigenvalues E_n.

   (b) [**3 points**] Using the momentum p as the summation index in Eq. (4), calculate the time evolution operator $U(x,t_2,x',t_1)$ for a free particle of mass m in one dimension. (Schwabl's Problem 2.1 and manipulations as in Eq. (2.8) may be useful.)

7. [**2 points**] Schwabl's Problem 4.1(a).

8. [**7 points**] The uncertainty relation is usually associated with quantum mechanics (it contains $\hbar$). At the same time, it can be rewritten without $\hbar$ as the uncertainty relation between wave vector and coordinate:

$$\Delta k\,\Delta x \geq 1/2, \tag{5}$$

or frequency and time:

$$\Delta\omega\,\Delta t \geq 1/2. \tag{6}$$

Eqs. (5) and (6) are well known in such classical disciplines as optics, electrical engineering, data transmission, etc. They are nothing but identities relating the width of a function and the width of its Fourier transform.

Show that computer circuits and music record players obey the uncertainty relation (6). Let us denote the voltage as a function of time t somewhere in the electric circuit by $f(t)$. For concreteness, let us assume that $f(t)$ has the shape of a pulse vanishing at $t \to \pm\infty$ sufficiently fast and is a real, normalized function:

$$\int f^2(t)\,dt = 1. \tag{7}$$

Arrival of the pulse signifies the arrival of a bit of information. However, since $f(t)$ has some width in time, there is an uncertainty in the arrival time. It is natural to define the center of the pulse as

$$t_0 = \int t\,f^2(t)\,dt. \tag{8}$$

Then, the uncertainty of the time of the signal is

$$\Delta t = \left(\int (t-t_0)^2 f^2(t)\,dt\right)^{1/2}. \tag{9}$$

The Fourier transform of the signal is

$$h(\omega) = \int e^{i\omega t} f(t) dt, \tag{10}$$

and the uncertainty of frequency is

$$\Delta\omega = \left(\int \omega^2 |h(\omega)|^2 \frac{d\omega}{2\pi}\right)^{1/2}. \tag{11}$$

With the definitions (9)–(11), prove Eq. (6) (**see Hints**). Generalize the proof to a complex function $f(t)$ with $f^2(t) \to |f(t)|^2$ in Eqs. (7)–(9) and $\omega^2 \to (\omega-\omega_0)^2$ in Eq. (11). Show that this proof solves Schwabl's Problem 4.2 in general form by identifying $f(t) = \psi(x,t)$ for a given x.

Eq. (6) states that, in order to reproduce short-time variations of a signal with high fidelity, the system must have sufficiently wide bandwidth in frequency. That is why a fiber optics data network, utilizing electromagnetic waves in the frequency range of visible light, can transmit digital data much faster than a copper wire network, whose range is limited to much lower frequencies.

Hints

8 Represent $\omega h(\omega)$ in Eq. (11) as

$$\omega h(\omega) = -i\int \frac{\partial e^{i\omega t}}{\partial t} f(t) dt, \tag{12}$$

integrate by parts in Eq. (12), and express $\Delta\omega$ in terms of $f(t)$.

Then, apply the Schwarz inequality (Eq. (4.1) of Schwabl, where the scalar product is understood in the integral sense (2.35)) to $(\Delta\omega)^2(\Delta t)^2$, and integrate by parts.