Phys 622, Introduction to Quantum Mechanics I, Fall 1998, Dr. Yakovenko

Final Examination

Saturday, December 19, 1998, 10:30 a.m.–12:30 p.m.

1. [**8 points**] A particle of mass m is constrained to move between two concentric impenetrable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground-state energy E_0 and normalized wave function $\psi_0(r)$.

2. [**8 points**] A system of three nonidentical particles of spin-1/2, whose spin operators are $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$, and $\hat{\mathbf{s}}_3$, is governed by the Hamiltonian

$$\hat{H} = A\,\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2/\hbar^2 + B\,(\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2) \cdot \hat{\mathbf{s}}_3/\hbar^2. \tag{1}$$

Find the energy levels and their quantum numbers and degeneracies.

3. [**8 points**] Three 23×23 Hermitian matrices $\hat{M}_x$, $\hat{M}_y$, and $\hat{M}_z$ are known to obey the commutation rules $[\hat{M}_x, \hat{M}_y] = i\hat{M}_z$ with all cyclic permutations of x, y, and z. The eigenvalues of $\hat{M}_y$ are $\pm 3/2$, each twice; ± 1, each 3 times; $\pm 1/2$, each 3 times; and 0, 7 times. State the 23 eigenvalues of the matrix $\hat{M}^2 = \hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2$ and their degeneracies.

4. [**8 points**] For a spinless system that is time-reversal invariant, prove that if an energy eigenstate is nondegenerate, the average momentum and orbital angular momentum vanish:

$$\langle \hat{\mathbf{p}} \rangle = 0, \qquad \langle \hat{\mathbf{L}} \rangle = 0. \tag{2}$$

(This is known as the *"quenching"* of orbital angular momentum, and it happens, for example, when an atom is subject to a low-symmetry crystal field.)

If a nondegenerate wave function is expanded as

$$\int d^3p\,\varphi(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \qquad \text{or} \qquad \sum_{lm} F_{lm}(r)Y_{lm}(\theta,\phi), \tag{3}$$

what kind of phase restrictions does the time-reversal symmetry impose on the coefficients $\varphi(\mathbf{p})$ and F_{lm}?

5. [**8 points**] Let us consider a hydrogen atom in semispace: The proton is located at the origin $x = y = z = 0$, and the electron can move only in the semispace $z > 0$, but cannot penetrate $z < 0$. Obviously the wave function of the electron must vanish at $z = 0$. Find the energies, wave functions, quantum numbers, and degeneracies of the ground state and all other bound states.

Directions: You don't need to solve any differential equations in this problem. Just exploit the symmetry of the problem to write the answer in terms of the known eigenenergies and wave functions $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$ of a regular hydrogen atom. Reminder: The parity of $Y_{lm}(\theta,\phi)$ is $(-1)^l$.