Exam 3—Solution—Phys374—Fall 2013 Prof. Alessandra Buonanno
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Each problem is worth 10 points.

1. In an extremely powerful explosion there is a rapid release of energy E in a small region of space, producing an outgoing spherical shock wave whose radius R grows with time t. Use dimensional analysis to determine how R depends on E, the initial mass density ρ_0 of the air, and t. Assume that those are the only relevant quantities.

   Solution: $E \sim ML^2T^{-2}$, $\rho_0 \sim ML^{-3}$, and $t \sim T$. Thus $Et^2 \sim ML^2$, and $Et^2/\rho_0 \sim L^5$. Therefore $R \propto (Et^2/\rho_0)^{1/5}$.

2. The Lennard-Jones potential for the interaction energy between two atoms separated by a distance r takes the form

$$V(r) = \frac{1}{12}r^{-12} - \frac{1}{6}r^{-6} \tag{1}$$

   when written in convenient units.

   (a) Find the r value $r_{\rm min}$ at the minimum of the potential.

   (b) Find the Taylor expansion of $V(r)$ around $r_{\rm min}$, keeping terms out through quadratic order in $r - r_{\rm min}$.

   (c) If the motion of a unit mass were governed by this potential, what would be the frequency of its small oscillations around $r_{\rm min}$?

   Solution:
   $V(r) = r^{-12}/12 - r^{-6}/6$
   $V'(r) = -r^{-13} + r^{-7}$
   $V''(r) = 13r^{-14} - 7r^{-8}$

   (a) $V'(r) = 0$ implies $r^6 = 1$, so $r_{\rm min} = 1$.

   (b) $V(r) = V(1) + V'(1)(r-1) + (1/2)V''(1)(r-1)^2 + \ldots = -(1/12) + 3(r-1)^2 + \ldots$

   (c) The potential near the minimum is that of a harmonic oscillator with spring constant $k = V''(1) = 6$. The angular frequency of oscillations for a mass $m = 1$ is $\omega = \sqrt{k/m} = \sqrt{6}$.

3. The *magnetic helicity* is a measure of the twisting of magnetic field lines around each other. It is given by an integral over all space, $\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B}\, dV$, where $\mathbf{A}$ is the vector potential and $\mathbf{B}$ is the magnetic field. These vector fields are related by $\mathbf{B} = \nabla \times \mathbf{A}$.

   (a) Show that if $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\lambda$ (a so-called "gauge transformation"), with λ any scalar function, $\mathbf{B}$ remains unchanged.

   (b) Show that the helicity is unchanged under a gauge transformation, assuming the magnetic field goes to zero sufficiently rapidly as the radius grows. (*Hint*: Integrate by parts using one of the vector calculus product rules, and use the fact that there are no magnetic monopoles.)

   FYI: For a perfectly conducting plasma, the helicity is a conserved quantity.

   Solution:
   (a) The key is that $\nabla \times \nabla\lambda$ vanishes identically for any (sufficiently differentiable) scalar field λ. (This identity holds because mixed partial derivatives commute. For example, the x component is $\partial_y \partial_z \lambda - \partial_z \partial_y \lambda \equiv 0$.) Thus

$$\begin{aligned} \mathbf{B}' &= \nabla \times (\mathbf{A} + \nabla\lambda) && (2)\\ &= \nabla \times \mathbf{A} + \nabla \times \nabla\lambda && (3)\\ &= \mathbf{B}. && (4) \end{aligned}$$

   (b) The key is the identity $\nabla\lambda \cdot \mathbf{B} = \nabla \cdot (\lambda\mathbf{B}) - \lambda\nabla \cdot \mathbf{B}$. The absence of magnetic monopoles implies $\nabla \cdot \mathbf{B} = 0$, hence $\nabla\lambda \cdot \mathbf{B} = \nabla \cdot (\lambda\mathbf{B})$. Thus

$$\begin{aligned} \mathcal{H}' &= \int (\mathbf{A} + \nabla\lambda) \cdot \mathbf{B}\, dV && (5)\\ &= \mathcal{H} + \int \nabla \cdot (\lambda\mathbf{B})\, dV && (6)\\ &= \mathcal{H} + \int_{\partial V} \lambda\mathbf{B} \cdot \mathbf{dS} && (7)\\ &= \mathcal{H}, && (8) \end{aligned}$$

   where the divergence theorem is used in the second to last step, and ∂V is the boundary of the volume, "at infinity", where $\mathbf{B} \to 0$ sufficiently rapidly for the integral to vanish.

4. Let $f(\theta)$ be the function that is given by 0 for $-\pi < \theta < 0$, and by $\sin\theta$ for $0 < \theta < \pi$, and satisfies $f(\theta + 2\pi) = f(\theta)$. Find all of the non-zero Fourier sine coefficients (don't worry about the cosine coefficients, even though they are nonzero). (*Hints*: (i) This is not complicated. (ii) You can relate the integral over $[0, \pi]$ to the one over $[-\pi, \pi]$, and then use a standard identity you proved in a homework problem.)

   Solution:

$$b_n = \pi^{-1} \int_0^{\pi} \sin\theta \sin(n\theta)\, d\theta \qquad (9)$$

$$= (1/2)\pi^{-1} \int_{-\pi}^{\pi} \sin\theta \sin(n\theta)\, d\theta \quad (10)$$

$$= (1/2)\delta_{1n}. \quad (11)$$

Thus only the $n = 1$ coefficient is nonzero.

5. The temperature $T(x,t)$ in an infinitely long, thin rod satisfies the heat equation

$$\partial_t T = \kappa \partial_x^2 T,$$

where $\kappa > 0$ is the heat conductivity. Assume that $T(x,t)$ may be expressed as a Fourier transform,

$$T(x,t) = \int \widetilde{T}(k,t)\, e^{ikx}\, dk. \quad (12)$$

(a) Insert (12) into the heat equation, and so doing find the differential equation satisfied by the Fourier transform $\widetilde{T}(k,t)$.

(b) Find the solution for $\widetilde{T}(k,t)$ in terms of its initial condition $\widetilde{T}(k,0)$ at time $t = 0$.

(c) Find $\widetilde{T}(k,0)$ for the case in which the initial temperature distribution is a Dirac delta function, $T(x,0) = A\delta(x-a)$, where A and a are constants.

FYI: Substituting these results for $\widetilde{T}(k,t)$ in (12), $T(x,t)$ becomes an explicit integral over k. This yields a Gaussian with center at $x = a$ and width proportional to $\sqrt{t}$.

Solution:

(a)

$$\partial_t T(x,t) = \int \partial_t \widetilde{T}(k,t)\, e^{ikx}\, dk \quad (13)$$

$$\kappa \partial_x^2 T(x,t) = \int \kappa (ik)^2 \widetilde{T}(k,t)\, e^{ikx}\, dk \quad (14)$$

Two functions of x are equal for all x if and only if their Fourier transforms are equal for all k, thus $\partial_t \widetilde{T}(k,t) = -\kappa k^2 \widetilde{T}(k,t)$.

(b) $\widetilde{T}(k,t) = e^{-\kappa k^2 t} \widetilde{T}(k,0)$.

(c)

$$\widetilde{T}(k,0) = (1/2\pi) \int T(x,0) e^{-ikx}\, dx \quad (15)$$

$$= (1/2\pi) \int A\delta(x-a) e^{-ikx}\, dx \quad (16)$$

$$= (A/2\pi) e^{-ika}. \quad (17)$$

$$(18)$$

FYI: This yields $T(x,t) = (A/2\pi) \int e^{-\kappa k^2 t} e^{i(x-a)k}\, dk = (A/\sqrt{4\pi\kappa t}) e^{-(x-a)^2/4\kappa t}$.

6. Evaluate the integral $\int_{-\infty}^{\infty} e^{-x}\,\delta(3+x^{-1})\,dx$.

Solution: If a function $g(x)$ is zero only at some x_0, then $\delta(g(x)) = |g'(x_0)|^{-1}\delta(x-x_0)$. In the present case $g(x) = 3 + x^{-1}$, so $x_0 = -1/3$, and $|g'(x_0)| = |-x_0^{-2}| = 9$. Hence the integral is equal to $e^{1/3}/9$

7. Two objects of mass m lie on a frictionless table, connected to each other with a spring constant k and connected to opposite walls with spring constant k for the mass on the left and $2k$ for the mass on the right. Consider only motions along a straight line.

   (a) Write the coupled equations of motion (Newton's second law) for the displacements x_1 and x_2 of the left and right masses from their equilibrium positions.

   (b) Find all the normal mode frequencies. How many are there?

   (c) Find the ratio x_2/x_1 of the displacements for the two masses in each of the normal modes. For each mode, state whether the masses move in the same direction or oppositely.

Solution: (a)

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2 \tag{19}$$
$$m\ddot{x}_2 = -k(x_2 - x_1) - 2kx_2 = kx_1 - 3kx_2 \tag{20}$$

(b) In matrix form, the equations are $\ddot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}.$$

The eigenvalues of A are given by the characteristic equation,

$$det(A - \lambda I) = (\lambda+2)(\lambda+3) - 1 = \lambda^2 + 5\lambda + 5 = 0,$$

in units with $k/m = 1$. The roots are $\lambda_\pm = (-5 \mp \sqrt{5})/2$. The frequencies are given by $\omega^2 = -\lambda$, i.e. $\omega_\pm = \sqrt{(5 \pm \sqrt{5})/2}\sqrt{k/m}$.

(c) The eigenvectors are determined by $(A - \lambda I)\mathbf{x} = 0$, i.e.

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -(\lambda+2)e_1 + e_2 \\ e_1 - (\lambda+3)e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $e_2 = (\lambda+2)e_1$, so $x_2/x_1 = e_2/e_1 = \lambda + 2 = (-1 \mp \sqrt{5})/2 = \{-1.618, 0.618\}$. The ω_- mode has the smaller frequency, and $x_2/x_1 = 0.618 > 0$, so the masses move in the *same* direction, with the second mass moving less. The ω_+ mode has the higher frequency, and $x_2/x_1 = -1.618$, so the masses move in *opposite* directions, with the second mass moving more.