Exam 2 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Tuesday, Nov. 19, 2013 Room 4212, (301)405-1440
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1. Consider the vector field $\mathbf{A} = 3y^2\,(\hat{\mathbf{x}} + \hat{\mathbf{z}})$. *[2+4+4=10 pts]*

   (a) Compute $\mathbf{B} = \nabla \times \mathbf{A}$.

   (b) Compute the flux of $\mathbf{B}$ outward through the five square faces of the cube $0 < x, y, z < L$, other than the $z = 0$ face. (*Hint*: To do it quickly, use Stokes' theorem.)

   (c) (i) Give a simple argument showing that $\mathbf{A}$ be expressed as the curl of another vector field $\mathbf{F}$, and (ii) give one such $\mathbf{F}$.

   Solution:

   (a) $\mathbf{B} = 6y\,(\hat{\mathbf{x}} - \hat{\mathbf{z}})$.

   (b) $\int_S \mathbf{B} \cdot \mathbf{dS} = \oint_{\partial \mathbf{S}} \mathbf{A} \cdot \mathbf{dl}$, where S is the surface composed of five faces, and ∂S is the square at the bottom. For the line integral around the square, the edges parallel to $\hat{\mathbf{y}}$ and the edge at $y = 0$ don't contribute. On the edge at $y = L$ from $x = L$ to $x = 0$ we have $\mathbf{A} \cdot \mathbf{dl} = -3L^2\,dl$ (the circulation around the square is counterclockwise in the xy plane since we are computing the *outward* flux), so it contributes $-3L^3$. (For the flux integral, the faces perpendicular to $\hat{\mathbf{y}}$ don't contribute, and the faces perpendicular to $\hat{\mathbf{x}}$ cancel each other because of opposite normal vectors. The face at the top, perpendicular to $\hat{\mathbf{z}}$, gives $\int(-6y)dxdy = -3L^3$ as well.)

   (c) (i) Since $\nabla \cdot \mathbf{A} = 0$, we know that $\mathbf{A} = \nabla \times \mathbf{F}$ for some $\mathbf{F}$, at least in a contractible region. (ii) The components of $\mathbf{F}$ must satisfy $A_x = 3y^2 = \partial_y F_z - \partial_z F_y$, $A_y = 0 = \partial_z F_x - \partial_x F_z$, $A_z = 3y^2 = \partial_x F_y - \partial_y F_x$. We can try for a solution with $F_y = 0$, a choice motivated since x and z derivatives need not play a role with a function only of y. The A_x component equation then suggests $F_z = y^3$ and the remaining equations are satisfied if $F_x = -y^3$. Thus $\mathbf{F} = y^3(-\hat{\mathbf{x}} + \hat{\mathbf{z}})$ satisfies $\mathbf{A} = \nabla \times \mathbf{F}$. To this $\mathbf{F}$ can be added the gradient of any scalar function.

2. Find (i) the Cartesian form and (ii) a polar form of $\frac{1+i}{(1-i)^2}$. *[2+3=5 pts]*

   Solution: (i)
$$\frac{1+i}{(1-i)^2} = \frac{1+i}{(1-i)^2}\frac{(1+i)^2}{(1+i)^2} = \frac{-1+i}{2} = -\tfrac{1}{2} + \tfrac{1}{2}i$$
   (ii) $z = re^{i\theta}$, with $r = 1/\sqrt{2}$ and $\tan\theta = -1$ with $y > 0$, so $\theta = 3\pi/4$. (An alternate way to compute: recognize that $1+i = \sqrt{2}e^{i\pi/4}$ and $1-i = \sqrt{2}e^{-i\pi/4}$, which immediately yields $(1+i)/(1-i)^2 = e^{i3\pi/4}/\sqrt{2}$.)

3. Find all complex numbers β for which $\cosh(z+\beta) = -\cosh(z)$ for all complex z. (*Hint*: Express cosh in terms of exponentials.) *[3 pts]*

Solution: $\cosh(z+\beta) = (e^{z+\beta} + e^{-z-\beta})/2 = -(e^z + e^{-z})/2$ implies $e^\beta = -1$, so $\beta = i\pi(2n+1)$ for any integer n.

4. State whether each of the following functions is analytic, and how you know. (i) $1/z$, (ii) $|z|$, (iii) $Re(z)$. (Exclude the point $z = 0$.) (iv) For what values of the complex number γ is $x^2 - 2\gamma xy - y^2$ an analytic function of $z = x + iy$, and why? *[3+3+3+3=12 pts]*

Solution: (i) analytic: function of z; (ii) not analytic: $|z| = zz*$ involves z^*. (iii) not analytic: $Re(z) = (z + z*)/2$ involves z^*. (iv) If the given function $h(x, y)$ is analytic, then the Cauchy-Riemann equation $\partial_y h = i\partial_x h$ holds, i.e. $-2y - 2\gamma x = i(2x - 2\gamma y)$ for all x and y. This requires $\gamma = -i$. (Note that then $h(x, y) = (x + iy)^2 = z^2$, i.e. h is purely a function of z.)

5. Consider a velocity potential given by the real part of $h(z) = A/z$, where A is a real positive constant. *[3+3+4=10 pts]*

   (a) Find the components (v_x, v_y) of the flow velocity as functions of x and y.

   (b) Find the equation for a general flow line in this flow.

   (c) Sketch the rough shape and location of the flow lines that go through the points $(x, y) = (\pm 1, 0)$ and $(x, y) = (0, \pm 1)$. Include arrows showing the direction of the flow. (The origin is a singular point of this flow.)

Solution:

(a) $h(z) = A/(x + iy) = A(x - iy)/(x^2 + y^2)$, so $f = Re[h] = Ax/(x^2 + y^2)$. The components of velocity are thus

$$v_x = \partial_x f = \frac{A}{x^2 + y^2} - \frac{Ax(2x)}{(x^2 + y^2)^2} = A\frac{y^2 - x^2}{(x^2 + y^2)^2} \quad (1)$$

$$v_y = \partial_y f = \frac{-2Axy}{(x^2 + y^2)^2} \quad (2)$$

(b) $Im[h] = const.$ implies $y/(x^2 + y^2) = C = const.$

(c) For a flow line through $y = 0$, the constant C is zero. The x axis thus coincides with a flow line. For the direction, note that for $y = 0$ we have $v_x = -A/x^2$, so the flow is towards negative x, and is infinitely fast at the origin. The flow line through $(x, y) = (0, \pm 1)$ has $C = \pm 1$, i.e. $\pm y = x^2 + y^2$, or better yet $x^2 = \pm y - y^2$. The two signs are related by reflection across the x axis, and each flow line is symmetric across the y axis. For the $y > 0$ case we have a loop that goes through the origin. The flow is clockwise around the loop: at $(0, 1)$ we have $v_x = A > 0$. (Also v_y is positive for $x < 0$ and negative for $x > 0$ in the $y > 0$ half plane.)

6. Evaluate the integral of $1/(1 + z^4)$ on the following two contours: *[4+6=10 pts]*
(a) a counterclockwise circle of radius 1/2 centered on the origin, (b) the positive real

axis from 0 to ∞. Be sure to fully justify all your steps and results. For part (b), show explicitly that your result is a positive real number.

Solution: (a) The poles are where $z^4 = -1$, i.e. at $z = e^{i\pi/4}$ times ± 1 or $\pm i$. These all lie on the unit circle, so the circle of radius 1/2 encloses no poles, hence the integral vanishes. (b) An arc at radius $R \gg 1$ contributes something of order $R/R^4 = 1/R^3$ which vanishes as $R \to \infty$, so an arc at infinity can be added to the contour without changing the integral. Along the imaginary axis we have $z = ir$, so $z^4 = r^4$ and $dz = idr$. That integral therefore yields $\int_\infty^0 (1 + r^4)^{-1}(idr)$, which is $-i$ times the original integral I. Thus $(1-i)I = \sqrt{2}e^{-i\pi/4} I$ is equal to the closed contour integral around the first quadrant. This encloses only the pole at $e^{i\pi/4}$, with residue $1/4z^3 = e^{-i3\pi/4}/4$. Thus $\sqrt{2}e^{-i\pi/4} I = 2\pi i(e^{-i3\pi/4}/4)$, so $I = \pi/2\sqrt{2}$.