HW#9 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Tuesday, Nov. 26, 2013 Room 4212, (301)405-1440
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1. Evaluate $\int_a^b \delta(x^2-3)dx$ for (i) $[a,b]=[-1,1]$, (ii) $[a,b]=[0,2]$, (iii) $[a,b]=[-2,0]$, (iv) $[a,b]=[-2,2]$. (See Chapter 14 for Dirac delta functions. Section 14.3 discusses delta function of a function, which was also explained in class.) [10 pts.]

   Solution: A delta function of a function $g(x)$ is a sum of deltas centered at the zeroes x_i of $g(x)$, i.e. $\delta(g(x)) = \sum_i |1/g'(x_i)|\,\delta(x-x_i)$. For $g(x)=x^2-3$ the zeroes are at $\pm\sqrt{3}$, and $g'(\pm\sqrt{3}) = \pm 2\sqrt{3}$, hence $\delta(x^2-3) = (1/2\sqrt{3})[\delta(x-\sqrt{3})+\delta(x+\sqrt{3})]$. Thus for the above integrals we have (i) 0, (ii) $1/2\sqrt{3}$, (iii) $1/2\sqrt{3}$, (iv) $1/\sqrt{3}$.

2. Consider the integral

$$I = \iint f(x,y)\delta(x^2+y^2-R^2)\delta((x-a)^2+y^2-R^2)\,dxdy,$$

   taken over the entire xy plane.

   (a) Make sketches in the (x,y) plane showing geometrically where the two delta functions in the integrand are non-zero, for $a/R = 0,1,2,3$.

   (b) Evaluate I. (*Suggestion*: First do the y integral, using the first delta function to identify the relevant y values.)

   (c) Explain the qualitative behavior of the dependence of I on a/R in terms of your sketch in part 2a. In particular explain why it diverges where it diverges, and where it is zero. (*Guidance*: Imagine the delta functions as having a small width, before taking the limit as the width goes to zero and the height to infinity, so each of their regions of nonzero support forms a ring. Consider how the area of the region in which both delta functions are non-zero depends on a/R. The idea behind this was explained in class.) [10 pts.]

   Solution:

   (a) The non-zero support of the first delta-function is on a circle of radius R centered on the origin, and for the second delta function it is a circle of radius R shifted by a to the right, i.e. centered on the point $(a,0)$. For $a=0$ these two circles coincide. For $a=R$ they intersect in two isolated points. For $a=2R$ they intersect at one point of tangency, and for $a=3R$ they do not intersect at all.

   (b)

$$I = \iint f(x,y)\delta(x^2+y^2-R^2)\delta((x-a)^2+y^2-R^2)\,dxdy \quad (1)$$

$$= \int_{-R}^{R} \left|2(R^2-x^2)^{1/2}\right|^{-1} \sum_{\pm} f\left(x, \pm(R^2-x^2)^{1/2}\right) \delta(-2ax+a^2)\, dx \quad (2)$$

$$= \theta(2R-|a|) \left|4a(R^2-(a/2)^2)^{1/2}\right|^{-1} \sum_{\pm} f\left(x, \pm(R^2-a^2/4)^{1/2}\right) \quad (3)$$

The step function appears in the last step since the second δ-function is zero unless $x = a/2$. This contributes only if $a/2$ falls within the limits of x integration, i.e. only if $|a/2| < R$.

(c) If $a = 0$ the integral is infinite, diverging as $1/a$. This is because the circles overlap. If each circle is replaced by a ring of width ϵ, then the value of the regulated delta scales as ϵ^{-1}, so the product of the two deltas scales as ϵ^{-2}. The area of the two overlapping rings scales as ϵ, so the integral blows up as ϵ^{-1}. If $a = R$, the intersection regions of the two rings have area that scales as ϵ^2, so we get a finite integral in the limit $\epsilon \to 0$. If $a = 2R$ the rings are tangent, and their overlapping area scales as $\epsilon^{3/2}$, so the integral diverges as $\epsilon^{-1/2}$. If $a = 3R$ the overlap area vanishes.

3. The relation between the real Fourier coefficients for the sine and cosine terms can be obtained with the help of the following identities:

$$\int_{-\pi}^{\pi} \cos(m\theta)\, \cos(n\theta)\, d\theta = \pi\delta_{mn} \quad (4)$$

$$\int_{-\pi}^{\pi} \sin(m\theta)\, \sin(n\theta)\, d\theta = \pi\delta_{mn} \quad (5)$$

$$\int_{-\pi}^{\pi} \cos(m\theta)\, \sin(n\theta)\, d\theta = 0, \quad (6)$$

where m and n are assumed to be positive integers. (These are equivalent to eqns (15.3-6) in the textbook.) Prove these identities by expressing the cosine and sin in terms of complex exponentials, and using $\int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = 2\pi\delta_{k0}$. ($\delta_{kl}$ is the Kronecker delta, equal to 1 if the integers k and l are equal, and zero otherwise). [10 pts.]

Solution: The first integrand is $(e^{im\theta}+e^{-im\theta})(e^{in\theta}+e^{-in\theta})/4 = (e^{i(m+n)\theta}+e^{-i(m+n)\theta}+e^{i(m-n)\theta}+e^{-i(m-n)\theta})/4$. The integral therefore gives $(\pi/2)(\delta_{m+n,0}+\delta_{-m-n,0}+\delta_{m-n,0}+\delta_{-m+n,0})$. Since m and n are positive, $m+n$ and $-m-n$ cannot be zero, so the first two terms give nothing. The last two Kronecker deltas give 1 when $m = n$, so the result is $\pi\delta_{mn}$. The analysis for the other cases is similar.