HW#5 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Oct. 17, 2013 Room 4212, (301)405-1440
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1. Derive the identity
$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\, \nabla \times \mathbf{v} \tag{1}$$
where f is a scalar field and $\mathbf{v}$ is a vector field. [5 pts.]

   Solution: This is a direct consequence of the product rule for derivatives, together with linearity of the derivative:
$$\begin{aligned} \nabla \times (f\mathbf{v}) &= [\partial_y(fV_z) - \partial_z(fV_y)]\hat{\mathbf{x}} + \cdots && (2)\\ &= [(\partial_y f)v_z - (\partial_z f)v_y)]\hat{\mathbf{x}} + f[\partial_y v_z - \partial_z v_y]\hat{\mathbf{x}} + \cdots && (3)\\ &= \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v} && (4)\end{aligned}$$

2. Evaluate the expression
$$\nabla \times (f(r)\mathbf{r}), \tag{5}$$
where $\mathbf{r}$ is the position vector from the origin to the point $\mathbf{r}$, and $r = |\mathbf{r}|$, using (i) Cartesian coordinates and (ii) spherical coordinates (cf. (7.16)). In the first step, use the result of the previous problem to simplify this one. [3+2=5 pts.]

   Solution:
$$\begin{aligned} \nabla \times (f(r)\mathbf{r}) &= (\nabla f) \times \mathbf{r} + f\nabla \times \mathbf{r} && (6)\\ &= f'(r)(\nabla r) \times \mathbf{r} + f\nabla \times \mathbf{r}. && (7)\end{aligned}$$

   Since $\nabla r = \hat{\mathbf{r}}$ the first term is proportional to $\mathbf{r} \times \mathbf{r} = 0$. (i) In Cartesian coordinates $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, so $\nabla \times \mathbf{r} = (\partial_y z - \partial_z y)\hat{\mathbf{x}} + \cdots = 0$, hence the second term vanishes. (ii) In spherical coordinates $\mathbf{r} = r\hat{\mathbf{r}}$, but only θ and φ derivatives of v_r enter (7.16), so $\nabla \times \mathbf{r} = 0$.

3. (i) Show that $\mathbf{F} = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$ has zero curl and divergence. Thus it can be written both as the gradient of a scalar and as the curl of a vector. (ii) Find all the scalar potentials for $\mathbf{F}$ (i.e. all functions Φ such that $\mathbf{F} = \nabla\Phi$). (iii) Find all the vector potentials for $\mathbf{F}$ (i.e. all vector fields $\mathbf{H}$ such that $\mathbf{F} = \nabla\times\mathbf{H}$). [5+5+5=15 pts.]

   Solution: (i) $\nabla\times\mathbf{F} = (\partial_y F_z - \partial_z F_y)\hat{\mathbf{x}} + \cdots = (\partial_y(xy) - \partial_z(zx))\hat{\mathbf{x}} + \cdots = (x-x)\hat{\mathbf{x}} + \cdots = 0$. (The y and z components are related by permutations of x,y,z so they need not be separately checked.) $\nabla \cdot \mathbf{F} = \partial_x(yz) + \partial_y(zx) + \partial_z(xy) = 0$.

   (ii) Since the curl of $\mathbf{F}$ vanishes, $\mathbf{F}$ can be written as the gradient of a scalar potential, i.e. $\mathbf{F} = \nabla\Phi$. This requires $\partial_x\Phi = F_x = yz$, $\partial_y\Phi = F_y = zx$, $\partial_z\Phi = F_z = xy$. Obviously a solution to this is given by $\Phi = xyz$. This is not the only solution, but if there is another solution Φ', i.e. $\mathbf{F} = \nabla\Phi'$, then $\nabla(\Phi - \Phi') = \mathbf{F} - \mathbf{F} = 0$. The only function with zero gradient everywhere is a constant, so Φ and Φ' differ at most by a constant. Hence the most general scalar potential is given by $\Phi = xyz + C$, where C is any constant. Alternatively, one can always find a scalar potential just by integrating along any path from some fixed reference point. For example let the path γ run along the coordinate axes as $(0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$, and define $f(x,y,z) = \int_\gamma \mathbf{F} \cdot \mathbf{dl}$. This will yield the same result as above. A different starting point for the integral would just add a constant.

(iii) Since the divergence of $\mathbf{F}$ vanishes, $\mathbf{F}$ can be written as the curl of a vector potential, i.e. $\mathbf{F} = \nabla \times \mathbf{H}$. If $\mathbf{H}'$ is also a vector potential for $\mathbf{F}$, then $\nabla \times (\mathbf{H} - \mathbf{H}') = 0$, which implies that $\mathbf{H} - \mathbf{H}' = \nabla f$ for some f. That is, two vector potentials differ by the gradient of a scalar function. Using this freedom we may simplify the search for a single vector potential. For example, the z component of the last equation reads $H_z - H_z' = \partial_z f$. Given any H_z', we can always find f such that $H_z' + \partial_z f = 0$. Thus, without loss of generality we may assume $H_z = 0$. Then $\mathbf{F} = \nabla \times \mathbf{H}$ becomes

$$yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}} = -\partial_z H_y\,\hat{\mathbf{x}} + \partial_z H_x\,\hat{\mathbf{y}} + (\partial_x H_y - \partial_y H_x)\,\hat{\mathbf{z}}. \quad (8)$$

The x component implies $H_y = -yz^2/2 + A(x,y)$ and the y component implies $H_x = xz^2/2 + B(x,y)$. The z component then implies $\partial_x A(x,y) - \partial_y B(x,y) = xy$, which can be satisfied for example by setting $B = 0$ and $A = x^2y/2$. Thus, $\mathbf{H} = (x^2 + z^2)y/2\,\hat{\mathbf{x}} - yz^2/2\,\hat{\mathbf{y}} + \nabla f$, where f is an arbitrary function. Alternatively, one can use the formulae in the following problem to find a vector potential.

4. Show that if $\mathbf{v}$ has vanishing divergence, then $\mathbf{v}$ is equal to the curl of $\mathbf{w}$ defined by the components

$$w_x(x,y,z) = \int_0^z dz'\, v_y(x,y,z') \quad (9)$$

$$w_y(x,y,z) = -\int_0^z dz'\, v_x(x,y,z') + \int_0^x dx'\, v_z(x',y,0) \quad (10)$$

$$w_z(x,y,z) = 0. \quad (11)$$

This proves by explicit construction that every divergenceless vector field can be expressed as the curl of another vector field. [5 pts.]

Solution:

$$(\nabla \times \mathbf{w})_x = \partial_y w_z - \partial_z w_y = v_x(x,y,z) \quad (12)$$

$$(\nabla \times \mathbf{w})_y = \partial_z w_x - \partial_x w_z = v_y(x,y,z) \quad (13)$$

$$(\nabla \times \mathbf{w})_z = \partial_x w_y - \partial_y w_x \quad (14)$$

$$= -\int_0^z dz'\,(\partial_x v_x + \partial_y v_y)(x,y,z') + v_z(x,y,0) \quad (15)$$

$$= \int_0^z dz'\,\partial_z v_z(x,y,z') + v_z(x,y,0) \quad (16)$$

$$= v_z(x,y,z). \quad (17)$$

5. This problem will illustrate Stokes' theorem on the surface of a capped cylinder. Using standard cylindrical coordinates (ρ, φ, z), consider the cylinder of radius $\rho = R$

running from $z = z_1$ to $z = z_2$. Let $\mathcal{S}$ be the surface of the cylinder together with the disk of radius R at $z = z_2$, so the boundary $\partial\mathcal{S}$ is just the circle of radius R at $z = z_1$. Choose the orientation on the circle consistent with the outward orientation on the cylinder. Assume a vector field of the form

$$\mathbf{v} = v_\varphi(\rho, z)\,\hat{\boldsymbol{\varphi}}. \tag{18}$$

In class we showed by integrating around an infinitesimal loop in the plane perpendicular to $\hat{\mathbf{z}}$ that $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}} = \rho^{-1}\partial_\rho(\rho v_\varphi)$. [5+3+2=10 pts.]

(a) Integrate around an infinitesimal loop in the plane perpendicular to $\hat{\boldsymbol{\rho}}$ to show that $(\nabla \times \mathbf{v}) \cdot \hat{\boldsymbol{\rho}} = -\partial_z v_\varphi$.

Solution: Consider the loop with corners (ρ, φ, z), $(\rho, \varphi + d\varphi, z)$, $(\rho, \varphi + d\varphi, z + dz)$, $(\rho, \varphi, z + dz)$. The loop integral, up to quadratic order in the differentials, is

$$[v_\varphi(\rho, \varphi, z) - v_\varphi(\rho, \varphi, z + dz)](\rho\, d\varphi) + [v_z(\rho, \varphi + d\varphi, z) - v_z(\rho, \varphi, z)]dz,$$

and using the definition of derivative this is equal to

$$(\rho^{-1}\partial_\varphi v_z - \partial_z v_\varphi)\rho d\varphi\, dz.$$

For a vector field with no φ dependence, only the second term survives. Dividing by the area $\rho d\varphi\, dz$ yields $(\nabla \times \mathbf{v}) \cdot \hat{\boldsymbol{\rho}} = -\partial_z v_\varphi$.

(b) Evaluate the integral $\int_{\mathcal{S}}(\nabla \times \mathbf{v}) \cdot \mathbf{dS}$ (don't use Stokes' theorem).

Solution:

$$\begin{aligned}
\int_{\mathcal{S}}(\nabla \times \mathbf{v}) \cdot \mathbf{dS} &= \int_{disk}(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}}\, dA + \int_{cylinder}(\nabla \times \mathbf{v}) \cdot \hat{\boldsymbol{\rho}}\, dA && (19)\\
&= \int_{disk}\rho^{-1}\partial_\rho(\rho v_\varphi)\,\rho d\rho d\varphi - \int_{cylinder}\partial_z v_\varphi\,\rho d\varphi\, dz && (20)\\
&= 2\pi R v_\varphi(R, z_2) - 2\pi R[v_\varphi(R, z_2) - v_\varphi(R, z_1)] && (21)\\
&= 2\pi R v_\varphi(R, z_1). && (22)
\end{aligned}$$

(c) Evaluate the integral $\oint_{\partial\mathcal{S}}\mathbf{v}\cdot\mathbf{dl}$ (don't use Stokes' theorem). According to Stokes' theorem, the result should agree with the part 5b.

Solution:

$$\begin{aligned}
\oint_{\partial\mathcal{S}}\mathbf{v} \cdot \mathbf{dl} &= \oint_{circle}\mathbf{v} \cdot \hat{\boldsymbol{\varphi}}\, dl && (23)\\
&= \oint_{circle} v_\varphi\,\rho\, d\varphi && (24)\\
&= 2\pi R v_\varphi(R, z_1). && (25)
\end{aligned}$$

6. *Quantized superfluid circulation and vortices*

A superfluid is a fluid whose microscopic thermal motions are strictly absent because of quantum mechanical behavior at low temperature. The velocity field of a superfluid composed of particles of mass m has the form $\mathbf{v} = (\hbar/m)\nabla\alpha$, where $\hbar$ is Planck's constant. The scalar function $\alpha(\mathbf{r})$ is actually the phase of a complex function $\psi(\mathbf{r}) = A(\mathbf{r})e^{i\alpha(\mathbf{r})}$ that describes the quantum state of the superfluid. [3+3+4=10 pts.]

(a) Show that the vorticity vanishes in a superfluid.

(b) Show that if α were a continuous function, the circulation would necessarily be zero on any loop, no matter what is inside the loop.

(c) Since α is by definition a *phase*, it need not be continuous. It is sufficient that $e^{i\alpha}$ be continuous. This allows α to have jumps that are an integer multiple of 2π. Show that this means that the circulation around any loop is "quantized", i.e. it can take on only a discrete set of values, labeled by an integer. What are these values?

Solution:

(a) $\boldsymbol{\omega} = \nabla \times \mathbf{v} = (\hbar/m)\nabla \times \nabla\alpha \equiv 0$.

(b) $\oint \mathbf{v}\cdot d\mathbf{l} = (\hbar/m)\oint \nabla\alpha\cdot d\mathbf{l} = (\hbar/m)(\alpha_2 - \alpha_1)$, where α_1 and α_2 are the values of α at the beginning and end of the loop respectively. If α is a continuous function these are equal, so the circulation vanishes.

(c) This part of the problem was not quite posed correctly. The point is not really that α can be discontinuous, but rather that, since it only appears in $e^{i\alpha}$, it is only *defined* up to addition of an integer multiple of 2π. As long as $u = e^{i\alpha}$ is differentiable, then within any interval we may choose a unique α that is also differentiable. However, on a closed loop, it is possible that no single α can be defined that is differentiable everywhere. For example, consider $u = e^{in\varphi}$, where φ is the azimuthal angle in cylindrical coordinates. In order not to be a "multiple-valued" function, φ must jump from 2π to 0, despite the fact that $e^{in\varphi}$ is differentiable at 2π. What this means for the circulation in a superfluid is that we must allow for the possibility that α differs at the end of the closed loop from its value at the beginning, but only by an integer multiple of 2π, i.e. $\alpha_2 - \alpha_1 = 2\pi n$. The circulation is then given by i.e. $\oint \mathbf{v}\cdot d\mathbf{l} = n2\pi\hbar/m$. Thus, although the vorticity vanishes in a superfluid, the the circulation need not vanish, but it is constrained to have one of these quantized values.

This problem shows that if the flow is in a toroidal region, for example between two nested cylinders, then the circulation around the torus can be nonzero but is quantized. A superfluid cannot have nonzero quantized circulation if it completely fills a simply connected region, for example inside a single cylinder, since any loop is then the boundary of a disc on which the vorticity vanishes everywhere, so the circulation must vanish too. However, superfluids can contain vortices in the core of which the fluid is not in the superfluid state. Then the vorticity need not vanish in the vortex core, so there may be nonzero circulation around the vortex. As in a toroidal region, the circulation around such vortices is quantized.