HW#4 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Oct. 3, 2013 Room 4212, (301)405-1440
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1. Problems 5.2a,b. (*Pressure gradient*) (For 5.2a, scan, photocopy, or trace the map.) For 5.2b, you'll have to work out or look up the distance across Ireland in the relevant direction.) [5+5=10 pts.]

   Solution:

   a. The gradient vector is perpendicular to the contours of constant pressure, points in the direction of increasing pressure, and has a length inversely proportional to the distance between neighboring contours.

   b. The pressure changes by $1030 - 1025 = 5$ millibars in what looks to me on a map of Ireland to be about 400 km, so the magnitude of the gradient is about 0.013 millibars/km. It is larger over Iceland since the pressure changes by the same amount in a shorter distance of about 200 km. Hence the gradient is about twice as large over Iceland as over Ireland.

2. Consider the function $f(x, y, z) = ax^2 + by^2 + cz^2$. [2+2+2+2+2=10 pts.]

   (a) Find ∇f.

   (b) Find the rate of change of f at the point $(1, 1, 1)$ in the direction of the position vector $\mathbf{r}$. (*Caution*: $\mathbf{r}$ is not a unit vector.)

   (c) Find the rate of change of f at the point $(1, 1, 1)$ in the direction of most rapid increase of f.

   (d) The level sets $f =$ const. are ellipsoids. Find the unit normal to the ellipsoid at the point $(1, 1, 1)$.

   (e) What is the angle between $\mathbf{r}$ and the normal to the ellipsoid at $(1, 1, 1)$? Check that in the spherically symmetric case $a = b = c$ the angle is zero.

   Solution: I'll use component notation.

   (a) $\nabla f = (2ax, 2by, 2cz)$.

   (b) At $\mathbf{r} = (1, 1, 1)$ we have $|\mathbf{r}| = \sqrt{3}$ so $\hat{\mathbf{r}} = (1, 1, 1)/\sqrt{3}$. Also $\nabla f = (2a, 2b, 2c)$. Hence the rate of change of f in the direction of the position vector is $\nabla f \cdot \hat{\mathbf{r}} = (2a, 2b, 2c) \cdot (1, 1, 1)/\sqrt{3} = (2/\sqrt{3})(a + b + c)$.

   (c) The gradient itself points in the direction of most rapid increase. The rate of change in this direction is just the magnitude of the gradient, $2\sqrt{a^2 + b^2 + c^2}$.

   (d) The gradient is normal to the level sets (surfaces of constant value) of a function. Hence the unit normal is the gradient divided by its norm, $\hat{\mathbf{n}} = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$.

   (e) The angle between $\mathbf{r}$ and the unit normal is the inverse cosine of $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = (1, 1, 1) \cdot (a, b, c)/(\sqrt{3}\sqrt{a^2 + b^2 + c^2}) = (a + b + c)/(\sqrt{3}\sqrt{a^2 + b^2 + c^2})$. This is equal to 1 when $a = b = c$, so the angle is zero.

3. Derive the following identities [2+2=4 pts]:

(a) If f is a scalar field and $\mathbf{v}$ is a vector field then

$$\nabla \cdot (f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f\nabla \cdot \mathbf{v}. \tag{1}$$

Solution: This is a direct consequence of the product rule for derivatives, together with linearity of the derivative:

$$\begin{aligned} \nabla \cdot (f\mathbf{v}) &= \partial_x(fv_x) + \partial_y(fv_y) + \partial_z(fv_z) && (2)\\ &= (\partial_x f)v_x + (\partial_y f)v_y + (\partial_z f)v_z + f(\partial_x v_x + \partial_y v_y + \partial_z v_z) && (3)\\ &= \nabla f \cdot \mathbf{v} + f\nabla \cdot v\,. && (4) \end{aligned}$$

(b) If f is a scalar field and h is a function of one variable, then

$$\nabla h(f) = h'(f)\nabla f. \tag{5}$$

Solution: Let's check the x component:

$$\Big(\nabla f(h)\Big)_x = \partial_x\Big(f(h)\Big) = f'(h)\partial_x h = f'(h)(\nabla h)_x.$$

4. In this problem r and $\mathbf{r}$ are the distance and the position vector from the origin. [20 pts.]

(a) (i) Show using both cartesian and spherical coordinates that $\nabla r = \hat{\mathbf{r}}$. (ii) Explain why this is dimensionally balanced. (iii) Derive this equation by a geometrical discussion of the properties of the direction and magnitude of ∇r. [(2+2)+2+2=8 pts.]

Solution: (i) Cartesian: $\nabla r = \partial_x r\,\hat{\mathbf{x}} + \cdots = (x/r)\,\hat{\mathbf{x}} + \cdots = (1/r)\mathbf{r} = \hat{\mathbf{r}}$. Spherical: $\nabla r = \partial_r r\,\hat{\mathbf{r}} = \hat{\mathbf{r}}$. (ii) Both sides are dimensionless: $\nabla \sim L^{-1}$ and $r \sim L$ so $\nabla r \sim 1$. On the other side, $\hat{\mathbf{r}} = (1/r)\mathbf{r} \sim L^{-1}L^1 \sim 1$ as well.

(b) Show that $\nabla \cdot \mathbf{r} = 3$. [2 pts.]

Solution: Using Cartesian coordinates, $\nabla \cdot \mathbf{r} = \partial_x x + \partial_y y + \partial_z z = 3$. Alternatively, using spherical coordinates, $\nabla \cdot \mathbf{r} = r^{-2}\partial_r(r^2 r) = 3$.

(c) Show that $\nabla \cdot \hat{\mathbf{r}} = 2/r$ by the following method: write $\hat{\mathbf{r}} = r^{-1}\mathbf{r}$, and use the results of problem 3 and problem 4b. [2 pts.]

Solution: $\nabla \cdot \hat{\mathbf{r}} = \nabla \cdot (r^{-1}\mathbf{r}) = -r^{-2}\nabla r \cdot \mathbf{r} + r^{-1}\nabla \cdot \mathbf{r} = -r^{-2}\hat{\mathbf{r}} \cdot \mathbf{r} + 3r^{-1} = -r^{-1} + 3r^{-1} = 2r^{-1}$.

(d) Show that if $\mathbf{m}$ is a constant vector, then (i) $\nabla(\mathbf{m} \cdot \mathbf{r}) = \mathbf{m}$ and (ii) $\nabla(\mathbf{m} \cdot \hat{\mathbf{r}}) = r^{-1}(\mathbf{m} - (\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})$. [2+2=4 pts.]

Solution: (i) $\nabla(\mathbf{m} \cdot \mathbf{r}) = \partial_x(m_x x + m_y y + m_z z)\hat{\mathbf{x}} + \cdots = m_x\hat{\mathbf{x}} + \cdots = \mathbf{m}$.
(ii) $\nabla(\mathbf{m} \cdot \hat{\mathbf{r}}) = \nabla(r^{-1}\mathbf{m} \cdot \mathbf{r}) = -r^{-2}\mathbf{m} \cdot \mathbf{r}\,\nabla r + r^{-1}\mathbf{m} = r^{-1}(\mathbf{m} - (\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})$.

(e) The magnetic field of a dipole moment $\mathbf{m}$ is $\mathbf{B}(\mathbf{r}) = (3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m})/r^3$. Use the above results to show that this has zero divergence (as does *any* magnetic field satisfying Maxwell's equations). (The textbook does this using the explicit cartesian components of $\mathbf{B}$ for the case $\mathbf{m} = m\hat{\mathbf{z}}$.) [4 pts.]

Solution: To make this more useful, and a little easier to follow, note first that $\hat{\mathbf{r}}\cdot\nabla(\mathbf{m}\cdot\hat{\mathbf{r}}) = 0$, so we can omit this term in evaluating $\nabla\cdot\mathbf{B}$. This is "obvious" once you think about it, since $\hat{\mathbf{r}}$ does not change along the $\hat{\mathbf{r}}$ direction. This is verified computationally, using the result of the previous part: $\nabla(\mathbf{m}\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = r^{-1}(\mathbf{m} - (\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = 0$. Thus

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \nabla\cdot[r^{-3}(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m})] && (6)\\
&= -3r^{-4}\hat{\mathbf{r}}\cdot(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}) + 3r^{-3}(\mathbf{m}\cdot\hat{\mathbf{r}})\nabla\cdot\hat{\mathbf{r}} && (7)\\
&= -6r^{-4}\mathbf{m}\cdot\hat{\mathbf{r}} + 6r^{-4}(\mathbf{m}\cdot\hat{\mathbf{r}}) && (8)\\
&= 0. && (9)
\end{aligned}$$

5. Problems 8.2a,b,c,d (*Gravitational field of a spherically symmetric mass*) For part (c) you may just sketch the graph. No careful plot is required. [2+2+4+2=10 pts.]

Solution: (a) Integrate both sides over a volume, and use Gauss' theorem to convert the volume integral of $\nabla\cdot\mathbf{g}$ to the surface integral in (8.4).

(b) With S a sphere of radius r the surface element is $\mathbf{dS} = \hat{\mathbf{r}}\,r^2 d\Omega$. With $\mathbf{g} = g(r)\hat{\mathbf{r}}$ we thus have $\mathbf{g}\cdot\mathbf{dS} = g(r)r^2 d\Omega$. Integrating over the sphere then yields $4\pi r^2 g(r) = -4\pi GM$, hence $g(r) = -GM/r^2$.

(c) Integrating over a sphere of radius $r < R$ yields (8.4) again, but in place of M we have the amount of the mass $M(r) = (r/R)^3 M$ lying inside the radius r: the fraction $(r/R)^3$ of the total volume times the total mass M. Thus we can just use (8.5), with the same replacement, yielding $g(r) = -GM(r/R)^3/r^2 = -GMr/R^3$. At $r = R$ this agrees with the result from part (b), $-GM/R^2$. The plot of $g(r)$ is a straight line with negative slope starting at 0 for $r = 0$ and decreasing linearly to $-GM/R^2$ at $r = R$. For $r > R$ the plot then rises, asymptotically approaching the axis.

(d) For the electric field the calculation is identical, apart from the constant factors. We get $\int_V \nabla\cdot\mathbf{E}\,\mathbf{dV} = \oint_S \mathbf{E}\cdot\mathbf{dS} = (1/\epsilon_0)\int_V \rho\,\mathbf{dV}$, which vanishes for a volume inside the hollow sphere. By spherical symmetry, we have $\mathbf{E} = E(r)\hat{\mathbf{r}}$, so the surface integral is $4\pi r^2 E(r)$. Since the surface integral vanishes, we infer that $E(r) = 0$, so the electric field vanishes.