HW#1 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Tuesday, Sep. 10, 2013 Room 4212, (301)405-1440
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1. **Scaling of cruising velocity of flying objects with respect to mass**: Problems 2.5d,e from from textbook. (You need not (but may) follow the steps and instructions in parts a,b,c,d. The book's method seems a bit inelegant to me. Note that eqns (2.30) and (2.31) should have $\sim$ signs rather than $=$ signs.)

   Solution: Part d requires little explanation, but I'll point out that the essential fact of the lift depending on the 6th power of the velocity can be obtained very simply with little fuss: At fixed air density and fixed flying object density we have from (2.27) $W \propto l^2 v^2$ where l is a linear dimension of the object. We also have that the weight is proportional to the mass which is proportional to the volume, so $W \propto l^3$ and hence $l^2 \propto W^{2/3}$. It follows that $W^{1/3} \propto v^2$, hence $W \propto v^6$. As for part e, the previous proportionality implies $\log W = 6 \log v + C$ for some "constant" C, so the slope of $\ln W$ as a function of $\log v$ is 6 when C is fixed. When you measure this slope you must take care to use the logarithms rather than W and v as the values, and also note that the linear scales on the two axes are not equal.

   Note that the "constant" C is in fact $\log(C_L^3/\rho_f^2) + \log(\rho^3/g^2)$, so is not really constant for all the objects plotted since they have different density ρ_f and different C_L. Larger density will shift a point downwards, and lower density will shift it upwards, but compared to the velocity dependence this effect and that of C_L is not so great. The human powered airplane seems to have the largest low density shift effect on the graph. If we had chosen to take the log of (2.27), we'd have obtained $\log W = 2 \log v + \log S + \log(C_L \rho)$. In this case the object-dependent term $\log S$ varies widely, since the surface area changes by many orders of magnitude between a crane fly, a Canada goose, and a Boeing 767. hence we would *not* obtain anything like a straight line with slope 2.

2. **Flow rate through a pipe**: Problems 2.6a,b,c from textbook. (*Hint for the last part*: the flow velocity is always zero at the walls...)

   Solution: The final question about why the flow rate depends on R^4 instead of R^2 is a bit obscure. The reason is that the flow rate scales as velocity times area, and the velocity in the pipe scales as R^2. Why R^2? Well, it is always zero at the walls and increases towards the center. It's not easy to see why it should scale precisely as R^2, however, without either using dimensional analysis or just computing it from the Hagen-Poiseuille equation.

3. **Ideal gas law**: Near the beginning of the nineteenth century Avogadro inferred from experiments the remarkable fact that the number of molecules N in a gas contained in a volume V at a pressure P and temperature T is the same for all gases. He deduced that the molecules must occupy only a very small fraction of the total volume. This led to the kinetic theory of gases, according to which the pressure arises from collisions between the molecules and the walls. Use dimensional analysis together with physical reasoning to determine the form of the function $N(V, P, T)$. Note that, up to an unknown coefficient, your result is just the ideal gas law!

   Notes: (i) Although it was not known at the time of Avogadro, make use of the fact that there is a universal constant k, *Boltzmann's constant*, such that kT has dimensions of energy. (kT is proportional to the average kinetic energy of a single molecule.) (ii) Since any function of a dimensionless quantity is dimensionless, you'll have to use something more than dimensional analysis to determine N. (*Hint*: Use physical reasoning to infer how N must depend on V at fixed P and T.)

   Solution: $V \sim L^3$, $P \sim ML^{-1}T^{-2}$, and $kT \sim ML^2T^{-2}$. To cancel the M factors we must take the ratio $P/kT \sim L^{-3}$, so the combination PV/kT is dimensionless. N must be a function of this combination. Using only dimensional analysis this is all that can be said. But the pressure and temperature are intensive (local) quantities, so at fixed P and T, N must be proportional to the extensive variable V. Thus $N \propto PV/kT$. (In fact, the dimensionless coefficient is exactly one, as seen from the ideal gas law $PV = NkT$.)

   Comment: If you double the size of the container and change nothing else, it seems clear that the number of molecules must double. However, note that this inference relies on the fact that molecules are small compared to the space between them, and that they have negligible interaction with each other. Indeed, for the the Van der Waals equation of state which takes into account the finite size and interaction between molecules, the number does not simply scale with the volume. However the corrections are small for a dilute gas. (See `en.wikipedia.org/wiki/Van_der_Waals_equation`.)

4. **Speed of surface waves on water**

The *phase velocity* of a wave is the speed of wavefronts of constant phase, while the group velocity is the speed of a wavepacket. Both have dimensions of velocity, of course, so dimensional analysis will not distinguish them.

The speed of water waves might depend upon various quantities: (i) properties of the water such as density ρ or surface tension (energy per unit area of stretching of a surface) σ, (ii) properties of the environment such as gravitational acceleration g or depth of the body of water h, and (iii) the wavelength of the wave λ. Depending on the wavelength of the wave different quantities are important in determining the wave speed. The speed is determined by the restoring force(s) that tend to return the surface to equilibrium, and the inertia of the water.

(a) First consider wavelengths long enough that surface tension can be neglected but short enough that the depth of the body of water can be neglected (*deep water waves*). (a) Use dimensional analysis to find how the wave speed depends on the remaining quantities ρ, g, λ. (b) Estimate (modulo the unknown dimensionless coefficient) the speed of a one meter wave on a lake.

Solution: $v \propto \sqrt{g\lambda}$, which for $\lambda = 1$ m is ~ 3 m/s. Note that in this regime longer wavelengths travel faster.

(b) Next consider wavelengths much longer than the depth (*shallow water waves*). Now the speed may depend upon the depth h as well, so there is a dimensionless ratio h/λ that the speed could depend on. However, one might reason that when this ratio is very small, the dependence on λ should drop out, since the restoring force is a local process and the concept of wavelength involves very distant parts of the wave. How then would the wave speed depend upon the various quantities ρ, g, h? Estimate (modulo the unknown dimensionless coefficient) the speed of a (long wavelength) tsunami on the Pacific ocean of depth 4 km.

Solution: $v \propto \sqrt{gh}$, which for $h = 4$ km is ~ 200 m/s $= 720$ km/hr. Note that in this regime all wavelengths travel at the same speed.

(c) Finally consider wavelengths short enough that the surface tension is the dominant restoring force and gravity can be neglected (*capillary waves*). How does the speed of capillary waves depend upon the remaining quantities ρ, λ, σ?

Solution: First $[\rho] = ML^{-3}$, $[\lambda] = L$, and $[\sigma] = [Energy]L^{-2}$. Then I'd remember that since kinetic energy is $1/2mv^2$, $[Energy] = ML^2T^{-2}$, so $[\sigma] = MT^{-2}$. Now you can write out arbitrary powers of ρ, λ, σ and solve for the powers to get a velocity, but personally I find it simpler to reason this way: velocity is LT^{-1} and among the three quantities only σ involves T, and that in the denominator, so we'll have $v \propto \sqrt{\sigma}$. But σ also involves M which is not in velocity, and this can only be cancelled by the M in ρ, so it must be $v \propto \sqrt{\sigma/\rho}$. However, $[\sigma/\rho] = L^3T^{-2}$, so we have to get rid of one extra L, which we can do by dividing by λ, viz $v \propto \sqrt{\sigma/\rho\lambda}$. Note that in this regime shorter wavelengths travel faster. This is why the ripples are bunched up at the leading edge of an expanding capillary wave disturbance on the surface of water.

(d) The phase velocity v_{ph} of a wave of angular frequency ω and wave vector k is ω/k, while the group velocity v_g of a wavepacket consisting of waves with wave vector near a given k is $d\omega/dk$. What is the exact ratio of group velocity to phase velocity for the three cases considered above? (Recall that $k = 2\pi/\lambda$.) (*Hint*: Consider the question for a dispersion relation of the general form $\omega = ck^n$ for some constant $c\ldots$)

Solution: If $\omega = ck^n$ for some constant c then $v_{ph} = ck^{n-1}$ and $v_g = nck^{n-1}$, so without knowing the constant c we can still say that $v_g/v_{ph} = n$, exactly! So we just need to find out what n is in the three cases. We can assume that the speed found in the three cases is either the group velocity or the phase velocity, because they are proportional to each other. Since $v \propto \omega/k$ we have $\omega \propto vk$. For deep waves, $v \propto \lambda^{1/2} \propto k^{-1/2}$, so $\omega \propto k^{1/2}$, so $n = 1/2$. The group velocity is half the phase velocity. For shallow waves $v \propto$ const., so $\omega \propto k$, so $n = 1$. The group and phase velocities are equal. For capillary waves $v \propto \lambda^{-1/2} \propto k^{1/2}$, so $\omega \propto k^{3/2}$, so $n = 3/2$. The group velocity is one and a half times the phase velocity.

(e) In the above cases, only the capillary wave speed depends on the density of the water. Explain physically why this is the case.

Solution: Only the capillary wave speed depends on the density. Why? A wave is a vibration, and all vibrations are determined by the inertia of the medium and the "restoring force" that pulls the medium back to its equilibrium configuration. So all three wave speeds should depend on the density of the water, since that determines the inertia. The reason why the deep and shallow wave speeds do not depend on the density is that the restoring force for these is gravity, which is *also* proportional to the inertial mass, so the mass density cancels out! It is for exactly the same reason that all objects fall with the same acceleration, and why the period of a pendulum is independent of the mass. It is this very strange property of gravity that is responsible: the famous equivalence of inertial and gravitational mass.