

Note written by Prof. Ted Jacobson

Complex numbers

1 Complex algebra and geometry

1. i as a solution to $x^2 + 1 = 0$, that is, $i = \sqrt{-1}$.
2. Complex numbers: $z = x + iy$, with x and y real numbers, called the real and imaginary parts of z , $x = \text{Re}(z)$ and $y = \text{Im}(z)$.
3. Square roots of other negative numbers. E.g. $\sqrt{-2} = i\sqrt{2}$.
4. Any quadratic equation $az^2 + bz + c = 0$ has two roots, given by $z = (-b \pm \sqrt{b^2 - 4ac})/2a$. If $b^2 - 4ac$ is negative, the roots are complex.
5. *Fundamental theorem of algebra*: Every polynomial has at least one root. This easily implies by induction that every n th order polynomial can be factorized as $a_n(z - w_1)(z - w_2) \cdots (z - w_n)$. That is, it has n roots, some or all of which may coincide. For a proof of this see below.
6. **Complex conjugate**: $z^* = x - iy$.
7. Inverse of a complex number:

$$\frac{1}{z} = \frac{z^*}{z^*z} = \frac{x - iy}{x^2 + y^2}. \quad (1)$$

8. **Modulus**: $|z| = \sqrt{z^*z} = \sqrt{x^2 + y^2}$.
9. Everything is nice: $(z + w)^* = z^* + w^*$, $(zw)^* = z^*w^*$, $(1/z)^* = 1/z^*$, $|zw| = |z||w|$, $|z^*| = |z|$.
10. **Exponential function**: The exponential function $f(x) = e^x$ can be defined as the unique function equal to its own derivative and equal to 1 when $x = 0$. To find its Taylor series, write $f(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and impose equality of df/dx and f for all x . The coefficients of x^n must then be equal, which implies $a_n = 1/n!$. Thus $e^x = \sum_{n=0}^{\infty} x^n/n!$. This definition can be generalized to any complex number z , replacing x by z .

11. **Key identity:** $e^a e^b = e^{a+b}$. One can prove this by writing out each of the series expansions, multiplying term by term on the left hand side, and expanding $(a + b)^n$ on the right hand side, then collecting terms with equal powers of a and b on both sides. A much simpler proof: define $f(a) = e^a e^b$ and $g(a) = e^{a+b}$, and note that $df/da = f$ and $dg/da = g$. Also $f(0) = g(0)$. So f and g are equal at one point and satisfy the same first order differential equation, so they are equal everywhere!
12. **Complex plane:** Complex numbers can be viewed as points or vectors on a plane, with $Re(z)$ on the horizontal axis and $Im(z)$ on the vertical axis. The distance from the origin to z is $|z|$.
13. **Euler's identity:** $e^{i\theta} = \cos \theta + i \sin \theta$. Two different proofs: (a) Expand both sides in a power series and show they are equal term by term, or (b) note that both sides are equal to i times their own derivative with respect to θ , so they satisfy the same first order differential equation. They are also obviously equal when $\theta = 0$, so they are equal everywhere.
14. $|e^{i\theta}| = 1$ for all real θ , so these complex numbers lie on the unit circle in the complex plane.
15. $e^{i\pi} + 1 = 0$, a remarkable equation, involving 0, 1, e , π and i .
16. **Polar form** of a complex number: $z = r e^{i\theta}$, where r and θ are the polar coordinates of z on the complex plane. The relation to the **Cartesian form** $z = x + iy$ is $x = r \cos \theta$ and $y = r \sin \theta$. Conversely, $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. There are two arctangents within the interval $\theta \in (-\pi, \pi)$; the correct one has the same sign as y . Conversely, $x = r \cos \theta$ and $y = r \sin \theta$. The angle θ is variously called the "angle" or "argument" of z , $Arg(z)$, or the "phase" of z . Sometimes $e^{i\theta}$ is called the phase.
17. **Geometric interpretations of conjugation and multiplication:** $z \rightarrow z^*$ is reflection across the real axis, and $z \rightarrow wz$ is scaling by $|w|$ and rotation counterclockwise by $Arg(w)$. In other words, when you multiply two complex numbers, *the moduli multiply and the angles add*:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}. \quad (2)$$

18. **i as a $\pi/2$ rotation:** Note that multiplication by i effects a counterclockwise $\pi/2$ rotation. Thus i^2 effects a counterclockwise π rotation... which is equivalent to multiplication by -1 . This gives a geometrical interpretation to the defining property of i , namely $i^2 = -1$.
19. **Logarithm:** $\ln z$ is any complex number such that $e^{\ln z} = z$. So $\ln z$ is *multi-valued*: any integer multiple of $2\pi i$ can be added. In terms of polar coordinates, $\ln z = \ln(re^{i\theta}) = \ln r + i(\theta + 2\pi n)$ where n is any integer.
20. **Complex powers of complex numbers:** z^w is *defined* to be $e^{w \ln z}$. This is in general multiple-valued, since $\ln z$ is. The multiple values are related by factors of $e^{w(i2\pi n)}$. If w is an integer, this is always 1. If w is rational, it can take on just a finite number of different values. For example, if $w = 1/2$, it is just ± 1 , the familiar sign ambiguity of the square root $z^{1/2}$.
21. **Caution!** There is a tricky point about the laws of complex exponents. Because of the multi-valuedness of the logarithm, in general $a^b a^c \neq a^{b+c}$, since $\ln a$ can be chosen differently in defining a^b and a^c . Moreover, depending on the values of a , b , and c , $a^b a^c$ may have more possible values than a^{b+c} . Similarly, $a^c b^c \neq (ab)^c$ and $(a^b)^c \neq a^{bc}$ in general. What's more, generally $\ln a^b \neq b \ln a$. *Exercise:* Find examples of these situations.
22. **Roots of unity:** $1^{1/n} = e^{\ln 1/n} = e^{i2\pi m/n}$, $m = 0, 1, \dots, n-1$. So there are n distinct n th roots of unity. Note that if w is an n th root of unity, then so is w^* .
23. Any complex number z has n distinct n th roots $z^{1/n}$, since any one can be multiplied by an n th root of unity. In polar form, $z^{1/n} = (re^{i\theta+i2\pi m})^{1/n} = r^{1/n} e^{i\theta/n+i2\pi m/n}$, with $m = 0, 1, \dots, n-1$
24. Example: $i^{1/3}$. Let's consider two different viewpoints. *Algebraically*, $i^{1/3} = e^{(\ln i)/3} = e^{i(\pi/2+2\pi n)/3} = e^{i(\pi/6+2\pi n/3)}$. In Cartesian form, the cube roots of i are $(\sqrt{3} + i)/2$, $(-\sqrt{3} + i)/2$, and $-i$. Check these by cubing them. *Graphically*, since multiplication by i is a $\pi/2$ counterclockwise rotation, $i^{1/3}$ can be $1/3$ of that rotation, i.e. a $\pi/6$ counterclockwise rotation, i.e. $e^{i\pi/6}$. But it can also be a $\pi/2$ clockwise rotation, i.e. $e^{-i\pi/2} = -i$. There is still one more possibility: one can multiply any value of $i^{1/3}$ by a cube root of unity and get another

value. The three cube roots of i are therefore spaced equally around a circle: $e^{i\pi/6}$, $e^{i5\pi/6}$, $e^{i9\pi/6} = -i$.

25. The fundamental theorem of algebra can be proved as follows. Let $p(z)$ be an n th order polynomial with nonzero constant term, i.e. $p(0) \neq 0$. Then $p(re^{i\theta})$ is a closed curve in the complex plane for each r . When $r = 0$ this curve is just the point $p(0)$. For sufficiently large r it is dominated by the highest order term in the polynomial, $a_n z^n$, so it is approximately $a_n r^n e^{in\theta}$, which follows a circle of radius $|a_n| r^n$ and loops around n times. As r decreases, this n -fold loop shrinks down, eventually to the point $p(0)$. On the way, each of the n turns of the loop must cross over the origin, and each $re^{i\theta}$ for which that happens is a root of the polynomial, i.e. $p(re^{i\theta}) = 0$. Generically these crossings are distinct, so there are n distinct roots. But for special polynomials one or more of these crossings can occur at the same value of $re^{i\theta}$. In that case the roots are degenerate. But there is always at least one root.

2 Complex calculus

1. **Analytic functions:** A complex function $h(z)$ is *analytic* if its derivative dh/dz is well-defined, i.e. exists and is independent of the direction dz takes in the complex plane. Other names for analytic functions are “regular” and “holomorphic”.
2. *Examples:* Any power of z is analytic, and so is any power series in z , or in $(z - z_0)$ for some fixed z_0 . The prototypical non-analytic function is $h(z) = z^* = x - iy$. Why? Well the derivative in the x direction is 1, while the derivative in the iy direction is $d(-iy)/d(iy) = -1$, so the two have opposite signs. More generally, if $dz = d(re^{i\varphi}) = e^{i\varphi} dr$, with φ fixed, then $dz^* = e^{-i\varphi} dr$, so $dz^*/dz = e^{-i2\varphi}$. So each direction for dz gives a different result, hence the function is not analytic. Other examples of non-analytic functions are $h(z) = (z + z^*)/2 = x$ and $h(z) = zz^*$.
3. **Analyticity and Taylor series:** If $h(z)$ is analytic in some region then in fact not just the first derivative but the derivatives of every order exist and are analytic in the region. Moreover, the Taylor series for $h(z)$ about any point z_0 in the region converges to $h(z)$ within a circle centered on z_0 that extends out to the nearest singularity of $h(z)$.

4. **Cauchy-Riemann equation:** The derivative dh/dz with $dz = dx$ is $\partial_x h$, while with $dz = d(iy) = idy$ it is $-i\partial_y h$ (since $1/i = -i$). Analyticity implies these two are necessarily equal,

$$\partial_y h = i\partial_x h \quad \text{complex form of Cauchy-Riemann equation} \quad (3)$$

Conversely, if $\partial_x h$ and $\partial_y h$ exist and are continuous in a region and satisfy the Cauchy-Riemann equation, then h is analytic in that region. In terms of the real and imaginary parts f and g of $h(z) = f(x) + ig(y)$, the Cauchy-Riemann equation becomes a pair of real equations,

$$\partial_x f = \partial_y g \quad \text{and} \quad \partial_y f = -\partial_x g. \quad (4)$$

5. If h is analytic then it is **harmonic**, i.e. it satisfies **Laplace's equation**:

$$\nabla^2 h \equiv \partial_x^2 h + \partial_y^2 h \quad (5)$$

$$= -i\partial_x \partial_y h + i\partial_y \partial_x h \quad (6)$$

$$= 0 \quad (7)$$

(using the Cauchy-Riemann equation (3) in the first step). This complex equation implies that the real and imaginary parts f and g are separately both harmonic. Conversely, if a real function is harmonic in a simply connected region, then it is the real (or imaginary) part of some analytic function in that region. This gives a powerful way to find harmonic functions.

6. If h is analytic then its gradient has zero norm:

$$\nabla h \cdot \nabla h = (\partial_x h)^2 + (\partial_y h)^2 \quad (8)$$

$$= (\partial_x h)^2 + i^2(\partial_x h)^2 \quad (9)$$

$$= 0. \quad (10)$$

The real and imaginary parts of this complex equation imply that the gradients of f and g have the same norm and are orthogonal:

$$|\nabla f| = |\nabla g| \quad \text{and} \quad \nabla f \cdot \nabla g = 0. \quad (11)$$

The second of these equations implies that the contours of f and g (i.e. the curves of constant f and g) intersect each other orthogonally. It also implies that the curves tangent to ∇f , i.e. the *flow lines* of ∇f , coincide with the contours of g , and *vice versa*.

7. **Contour integration:** $\int_C h(z) dz$, where C is a contour (curve) in the complex plane. This is defined in the usual way an integral is defined, as a limit of sums. It can also be expressed as a line integral of the complex vector field $\mathbf{H} = h\hat{\mathbf{x}} + ih\hat{\mathbf{y}}$:

$$\int h dz = \int h(dx + idy) \quad (12)$$

$$= \int (hdx + ihdy) \quad (13)$$

$$= \int \mathbf{H} \cdot d\mathbf{r}. \quad (14)$$

8. If h is analytic the vector field \mathbf{H} introduced in the previous item has vanishing curl and divergence, as a consequence of the Cauchy-Riemann equation: Since we are living on the x - y plane here, $(\nabla \times \mathbf{H})$ has only one component: $\partial_x H_y - \partial_y H_x = \partial_x(ih) - \partial_y h = 0$ by the complex Cauchy-Riemann equation. Similarly $\nabla \cdot \mathbf{H} = \partial_x H_x + \partial_y H_y = \partial_x h + \partial_y(ih) = 0$.
9. **Cauchy's theorem:** The integral $\oint h(z) dz$ around a closed contour vanishes if $h(z)$ is analytic in the enclosed region. This follows from writing the contour integral as the line integral of \mathbf{H} and converting it to the surface integral of $\nabla \times \mathbf{H}$ using Stokes' theorem, and then invoking the fact that for analytic h the vector field \mathbf{H} has vanishing curl.
10. **"Deforming the contour":** If the integrand is analytic in the region between an original contour and a continuously deformed contour, then Cauchy's theorem implies that the two contour integrals are equal.
11. **Key example:** Consider the integral $\oint z^n dz$ around a counterclockwise circle of radius r , where n is any integer. Cauchy's theorem tells us that this integral vanishes if $n \geq 0$, since then the integrand is analytic. But it also vanishes if $n < 0$, except if $n = -1$. This fact is central to the evaluation of contour integrals, and is simple to prove: On the circle we have $z = re^{i\theta}$ and $dz = izd\theta$, so

$$\oint z^n dz = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta \quad (15)$$

$$= 2\pi i \delta_{n,-1}, \quad (16)$$

where the *Kronecker delta* $\delta_{n,-1}$ is unity if $n = -1$ and vanishes otherwise. If the circle is deformed to any other closed contour without

crossing over the singularity at $z = 0$, the value of the integral remains unchanged. This key example is easily generalized to apply when the singularity is located somewhere other than $z = 0$.

12. **Poles and Residues:** Consider functions $h(z)$ that have an expansion

$$h(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,$$

in *negative and positive* integer powers of $(z - z_0)$, that is convergent for sufficiently small $|z - z_0|$. This is called a **Laurent expansion**. If a_n is non-zero for any negative n then $h(z)$ is singular at z_0 . If the number of such nonzero a_n is finite the singularity is called a *pole of order $|m|$* , where m is the highest negative power of $(z - z_0)$. A pole of order 1 is called a *simple pole*. The coefficient a_{-1} of the $(z - z_0)^{-1}$ term in the expansion is called the *residue* $Res h(z_0)$ of $h(z)$ at the pole z_0 .

13. **Residue theorem:** The integral of $h(z)$ counterclockwise around a closed contour enclosing poles at points z_i is given by

$$\oint h(z) dz = 2\pi i \sum_i Res h(z_i). \quad (17)$$

This generalizes the key example given above.

14. **Computing residues:** I'll discuss a few example situations. The simplest case occurs if $h(z) = g(z)/(z - z_0)$ with $g(z)$ analytic and nonzero at z_0 . Then $h(z)$ has a simple pole at z_0 and the residue is just $g(z_0)$. For example, the residue of $e^z/(z + 2)$ at $z = -2$ is e^{-2} . In terms of $h(z)$, we can write this as $Res h(z_0) = \lim_{z \rightarrow z_0} (z - z_0)h(z)$.

A slightly more involved case occurs if instead $h(z) = f(z)/g(z)$ has a simple pole at z_0 , where again $f(z)$ is analytic and nonzero at z_0 . Then, since $g(z) = g'(z_0)(z - z_0) + O((z - z_0)^2)$, the residue is $f(z_0)/g'(z_0)$. (Since we assumed z_0 is a simple pole, $g'(z_0) \neq 0$.) This is often a convenient, applicable rule. For example, the residue of $(z^3 + 1)^{-1}$ at one of its three simple poles (which occur at the cube roots of -1) is $1/(3z^2)$ evaluated at the pole. For instance, the residue at $z = e^{i\pi/3}$ is $(3e^{i2\pi/3})^{-1}$.

A more complicated situation arises if $h(z)$ can be expressed in the form $h(z) = f(z)/(z - z_0)^n$, where $f(z)$ is analytic and nonzero at z_0 .

Then $h(z)$ has a pole of order n at z_0 . The residue is the coefficient of the $(n-1)$ st term in the Taylor series of $f(z)$ about z_0 , $f^{(n-1)}(z_0)/(n-1)!$. For example, $z^3/(z-3)^3$ has a pole of order 3 at $z=3$, where the residue is $(1/2)(z^3)'' = 3z$ evaluated at $z=3$, i.e. 9.

In terms of $h(z)$, the residue in this last case can be expressed as

$$\text{Res } h(z_0) = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z-z_0)^n h(z)]. \quad (18)$$

In this form the result applies even if $h(z)$ is not presented in the form $f(z)/(z-z_0)^n$ with $f(z_0) \neq 0$. For example consider $(\sin z)/z^4$. The numerator is analytic at $z=0$, but it vanishes there, so this function does not have a pole of order 4 as it might at first seem to. Actually it has a pole of order 3 at $z=0$,

$$\frac{\sin z}{z^4} = \frac{z - z^3/3! + z^5/5! - \dots}{z^4} = \frac{1}{z^3} - \frac{1}{3!} \frac{1}{z} + \frac{1}{5!} z + \dots$$

The residue at $z=0$ is $-1/3!$. Here I found the residue by inspecting the expansion, of the numerator. It is equivalent to using the above formula, but use of the formula without making the series expansion of the sin would be more complicated, since you'd have to take the third derivative of $(\sin z)/z$ and then evaluate it at $z=0$.

15. **“Closing the contour”**: Often a real integral (or a contour integral over an open contour) can be related into a closed contour integral in the complex plane. In this case, the integral may be evaluated using the residue theorem. In extending an open contour to a closed contour one must add pieces to the contour. Sometimes an extra piece can be seen to contribute zero, for instance because it may be at infinity and the integrand may fall off rapidly enough for the integral over that part of the contour to vanish (even if that part of the contour is infinitely long). Sometimes two extra pieces can be seen to cancel each other. Sometimes an extra piece is nonzero, but is related by a multiplicative constant factor to the original integral. Then one can solve for the original integral in terms of the closed contour integral.