Exam 3—Phys374—Fall 2013 Prof. Alessandra Buonanno
Monday, Dec. 16, 2013 Room 4212, (301)405-1440
http://www.physics.umd.edu/rgroups/grt/buonanno/Phys374/ buonanno@umd.edu

Each problem is worth 10 points.

1. In an extremely powerful explosion there is a rapid release of energy E in a small region of space, producing an outgoing spherical shock wave whose radius R grows with time t. Use dimensional analysis to determine how R depends on E, the initial mass density ρ_0 of the air, and t. Assume that those are the only relevant quantities.

2. The Lennard-Jones potential for the interaction energy between two atoms separated by a distance r takes the form

$$V(r) = \frac{1}{12}r^{-12} - \frac{1}{6}r^{-6} \tag{1}$$

when written in convenient units.

   (a) Find the r value $r_{\rm min}$ at the minimum of the potential.

   (b) Find the Taylor expansion of $V(r)$ around $r_{\rm min}$, keeping terms out through quadratic order in $r - r_{\rm min}$.

   (c) If the motion of a unit mass were governed by this potential, what would be the frequency of its small oscillations around $r_{\rm min}$?

3. The *magnetic helicity* is a measure of the twisting of magnetic field lines around each other. It is given by an integral over all space, $\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B}\, dV$, where $\mathbf{A}$ is the vector potential and $\mathbf{B}$ is the magnetic field. These vector fields are related by $\mathbf{B} = \nabla \times \mathbf{A}$.

   (a) Show that if $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\lambda$ (a so-called "gauge transformation"), with λ any scalar function, $\mathbf{B}$ remains unchanged.

   (b) Show that the helicity is unchanged under a gauge transformation, assuming the magnetic field goes to zero sufficiently rapidly as the radius grows. (*Hint*: Integrate by parts using one of the vector calculus product rules, and use the fact that there are no magnetic monopoles.)

   FYI: For a perfectly conducting plasma, the helicity is a conserved quantity.

4. Let $f(\theta)$ be the function that is given by 0 for $-\pi < \theta < 0$, and by $\sin\theta$ for $0 < \theta < \pi$, and satisfies $f(\theta + 2\pi) = f(\theta)$. Find all of the non-zero Fourier sine coefficients (don't worry about the cosine coefficients, even though they are nonzero). (*Hints*: (i) This is not complicated. (ii) You can relate the integral over $[0, \pi]$ to the one over $[-\pi, \pi]$, and then use a standard identity you proved in a homework problem.)

5. The temperature $T(x,t)$ in an infinitely long, thin rod satisfies the heat equation

$$\partial_t T = \kappa \partial_x^2 T,$$

where $\kappa > 0$ is the heat conductivity. Assume that $T(x,t)$ may be expressed as a Fourier transform,

$$T(x,t) = \int \widetilde{T}(k,t)\, e^{ikx}\, dk. \tag{2}$$

   (a) Insert (2) into the heat equation, and so doing find the differential equation satisfied by the Fourier transform $\widetilde{T}(k,t)$.

   (b) Find the solution for $\widetilde{T}(k,t)$ in terms of its initial condition $\widetilde{T}(k,0)$ at time $t = 0$.

   (c) Find $\widetilde{T}(k,0)$ for the case in which the initial temperature distribution is a Dirac delta function, $T(x,0) = A\delta(x-a)$, where A and a are constants.

FYI: Substituting these results for $\widetilde{T}(k,t)$ in (2), $T(x,t)$ becomes an explicit integral over k. This yields a Gaussian with center at $x = a$ and width proportional to $\sqrt{t}$.

6. Evaluate the integral $\int_{-\infty}^{\infty} e^{-x}\, \delta(3 + x^{-1})\, dx$.

7. Two objects of mass m lie on a frictionless table, connected to each other with a spring constant k and connected to opposite walls with spring constant k for the mass on the left and $2k$ for the mass on the right. Consider only motions along a straight line.

   (a) Write the coupled equations of motion (Newton's second law) for the displacements x_1 and x_2 of the left and right masses from their equilibrium positions.

   (b) Find all the normal mode frequencies. How many are there?

   (c) Find the ratio x_2/x_1 of the displacements for the two masses in each of the normal modes. For each mode, state whether the masses move in the same direction or oppositely.