HW#8 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Nov. 14, 2013 Room 4212, (301)405-1440
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Note: *Be sure to fully justify the relation between the given integral and the closed contour integral you employ in the following problems.*

1. Find the residues of the following functions at the given values of z [$3 \times 5 = 15$ pts.]:
   (a) $(z + z^2)^{-1}$ at 0 and at -1.
   (b) $z^{-2} \ln(1 + 2z)$ at 0
   (c) $[z^3(z + 2)^2]^{-1}$ at 0 and at -2.
   (d) $\cos z/(2z - \pi)^4$ at $\pi/2$
   (e) $(z^2 + 1)^{-3}$ at $\pm i$.

2. (a) Show using contour integration that

$$\int_0^\infty \frac{\cos mx \, dx}{x^2 + a^2} = \frac{\pi}{2a} e^{-ma}$$

   Hint: See the similar example in the textbook.

   (b) Explain in words why the result decays so rapidly (i) as m grows with a fixed, and (ii) as a grows with m fixed. [8+2=10 pts.]

3. (a) Evaluate the integral $\int_0^\infty dx/(x^n + 1)$, where $n \geq 2$ is a positive integer, by relating it to the contour integral around the boundary of an infinite piece of pie with edges $\theta = 0$ and $\theta = 2\pi/n$, together with the arc at infinity that joins these edges. (b) Show that the result approaches 1 as $n \to \infty$, and explain with reference to the behavior of the integrand why this is the limiting value. (*Answer*: $(\pi/n)/\sin(\pi/n)$.) [8+2=10 pts.]

4. Consider potential flow perpendicular to an infinite solid cylinder of radius R. This reduces to a two-dimensional problem in the xy plane. The cylinder intersects the plane in a disk. For the boundary condition "at infinity", suppose that far from the cylinder in all directions the velocity is $\mathbf{v} = v_0\hat{\mathbf{x}}$. The boundary condition at the cylinder surface is that there is no flow perpendicular to the cylinder, so $\mathbf{v} \cdot \hat{\mathbf{r}} = 0$, taking the origin of polar coordinates at the center of the cylinder. That is, the partial derivative of the velocity potential with respect to radius r (at fixed angle) vanishes at $r = R$. To solve for the flow we need only find an analytic function of $z = x + iy$ whose real part satisfies the appropriate boundary conditions.

   (a) Find an analytic function $h_1(z)$ whose real part is a potential for the velocity field $\mathbf{v} = v_0\hat{\mathbf{x}}$.

   (b) Find an analytic function $h_2(z)$, to be added to your function $h_1(z)$ from the previous part, such that the real part of $h(z) = h_1(z) + h_2(z)$ satisfies the boundary condition everywhere on the cylinder, as well as the boundary condition at infinity. To do this assume $h_2(z) = az^n$, and find the values for constants a and

n for which the boundary conditions are satisfied everywhere on the cylinder. (*Hint*: Use polar coordinates. The result, which you are supposed to derive, is $h(z) = v_0(z + R^2/z)$.)

(c) Using results from Homework 7, find the velocity (magnitude and direction) at the point $(x, y) = (0, R)$ on the surface of the cylinder. How does it compare with v_0?

(d) Find the equation for the flow line that goes through the point $(x, y) = (0, y_0)$ (with $y_0 > R$). (The equation should involve x, y, y_0, v_0 and R.)

(e) Find the asymptotic y value y_∞ when $x \to \infty$ for the flow line of problem 4d.

(f) Sketch the flow lines of problem 4d for the cases $y_0 = R$ and $y_0 = 2R$, along with the circle of radius R representing the cross-section of the cylinder.