HW#5 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Oct. 17, 2013 Room 4212, (301)405-1440
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1. Derive the identity
$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\,\nabla \times \mathbf{v} \tag{1}$$
where f is a scalar field and $\mathbf{v}$ is a vector field. [5 pts.]

2. Evaluate the expression
$$\nabla \times (f(r)\mathbf{r}), \tag{2}$$
where $\mathbf{r}$ is the position vector from the origin to the point $\mathbf{r}$, and $r = |\mathbf{r}|$, using (i) Cartesian coordinates and (ii) spherical coordinates (cf. (7.16)). In the first step, use the result of the previous problem to simplify this one. [3+2=5 pts.]

3. (i) Show that $\mathbf{F} = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$ has zero curl and divergence. Thus it can be written both as the gradient of a scalar and as the curl of a vector. (ii) Find all the scalar potentials for $\mathbf{F}$ (i.e. all functions Φ such that $\mathbf{F} = \nabla\Phi$). (iii) Find all the vector potentials for $\mathbf{F}$ (i.e. all vector fields $\mathbf{H}$ such that $\mathbf{F} = \nabla \times \mathbf{H}$). [5+5+5=15 pts.]

4. Show that if $\mathbf{v}$ has vanishing divergence, then $\mathbf{v}$ is equal to the curl of $\mathbf{w}$ defined by the components
$$w_x(x,y,z) = \int_0^z dz'\, v_y(x,y,z') \tag{3}$$
$$w_y(x,y,z) = -\int_0^z dz'\, v_x(x,y,z') + \int_0^x dx'\, v_z(x',y,0) \tag{4}$$
$$w_z(x,y,z) = 0. \tag{5}$$
This proves by explicit construction that every divergenceless vector field can be expressed as the curl of another vector field. [5 pts.]

5. This problem will illustrate Stokes' theorem on the surface of a capped cylinder. Using standard cylindrical coordinates (ρ, φ, z), consider the cylinder of radius $\rho = R$ running from $z = z_1$ to $z = z_2$. Let $\mathcal{S}$ be the surface of the cylinder together with the disk of radius R at $z = z_2$, so the boundary $\partial\mathcal{S}$ is just the circle of radius R at $z = z_1$. Choose the orientation on the circle consistent with the outward orientation on the cylinder. Assume a vector field of the form
$$\mathbf{v} = v_\varphi(\rho, z)\,\hat{\boldsymbol{\varphi}}. \tag{6}$$
In class we showed by integrating around an infinitesimal loop in the plane perpendicular to $\hat{\mathbf{z}}$ that $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}} = \rho^{-1}\partial_\rho(\rho v_\varphi)$. [5+3+2=10 pts.]

   (a) Integrate around an infinitesimal loop in the plane perpendicular to $\hat{\boldsymbol{\rho}}$ to show that $(\nabla \times \mathbf{v}) \cdot \hat{\boldsymbol{\rho}} = -\partial_z v_\varphi$.

   (b) Evaluate the integral $\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathbf{dS}$ (don't use Stokes' theorem).

(c) Evaluate the integral $\oint_{\partial\mathcal{S}} \mathbf{v}\cdot\mathbf{dl}$ (don't use Stokes' theorem). According to Stokes' theorem, the result should agree with the part 5b.

6. *Quantized superfluid circulation and vortices*

A superfluid is a fluid whose microscopic thermal motions are strictly absent because of quantum mechanical behavior at low temperature. The velocity field of a superfluid composed of particles of mass m has the form $\mathbf{v} = (\hbar/m)\nabla\alpha$, where $\hbar$ is Planck's constant. The scalar function $\alpha(\mathbf{r})$ is actually the phase of a complex function $\psi(\mathbf{r}) = A(\mathbf{r})e^{i\alpha(\mathbf{r})}$ that describes the quantum state of the superfluid. [3+3+4=10 pts.]

(a) Show that the vorticity vanishes in a superfluid.

(b) Show that if α were a continuous function, the circulation would necessarily be zero on any loop, no matter what is inside the loop.

(c) Since α is by definition a *phase*, it need not be continuous. It is sufficient that $e^{i\alpha}$ be continuous. This allows α to have jumps that are an integer multiple of 2π. Show that this means that the circulation around any loop is "quantized", i.e. it can take on only a discrete set of values, labeled by an integer. What are these values?

This problem shows that if the flow is in a toroidal region, for example between two nested cylinders, then the circulation around the torus can be nonzero but is quantized. A superfluid cannot have nonzero quantized circulation if it completely fills a simply connected region, for example inside a single cylinder, since any loop is then the boundary of a disc on which the vorticity vanishes everywhere, so the circulation must vanish too. However, superfluids can contain vortices in the core of which the fluid is not in the superfluid state. Then the vorticity need not vanish in the vortex core, so there may be nonzero circulation around the vortex. As in a toroidal region, the circulation around such vortices is quantized.