**Phys 675**
**Homework problem S9.4**
University of Maryland, College Park, Fall 2014, Ted Jacobson

Here's a model of a rope hanging above a black hole. Suppose its stress-energy tensor has the form $T^\mu{}_\nu = \frac{1}{\Delta\Omega\, r^2}\text{diag}\{-\mu, -T, 0, 0\}$, where $\Delta\Omega$ is some fixed solid angle, outside of which the stress-energy tensor vanishes. So $\mu$ is the energy density per unit radial length and $T$ is the radial tension.

1. Show that if $T^{\mu\nu}$ is a symmetric tensor its divergence can be written as

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}}(\sqrt{-g}T^\mu{}_\nu)_{,\mu} - \frac{1}{2}g_{\alpha\beta,\nu}T^{\alpha\beta} \tag{1}$$

2. Apply part (a) to evaluate the $\nu = r$ component of the equation $\nabla_\mu T^\mu{}_\nu = 0$ for the rope model. Write the metric in the form $ds^2 = -\chi^2 dt^2 + \chi^{-2}dr^2 + r^2 d\Omega^2$. (For Schwarzschild, $\chi^2 = 1 - 2M/r$.) In this way, derive a first order ODE involving $\mu$, $T$, and $\chi$.

3. Find the general solution for the tension if the rope is massless, $\mu = 0$. What is the tension at infinity? As the horizon is approached?

4. Find all the solutions for $T$ if $\mu =$ constant. Can any of these support a rope that extends all the way to the horizon?

5. Show that $T = \mu =$ constant is a solution of the equation. Thus a "rope" of this sort can have a constant tension and reach all the way to the horizon of a black hole.

6. Suppose $T/\mu = w$ where $w$ is a positive constant less than 1, and find all the solutions. Can they extend all the way to the horizon?