In an expanding universe, what doesn't expand?

Richard H. Price
Department of Physics & Astronomy, Center for Gravitational Wave Astronomy, and Center for Advanced Radio Astronomy, University of Texas at Brownsville, Brownsville, Texas 78520

Joseph D. Romano
Department of Physics & Astronomy and Center for Gravitational Wave Astronomy, University of Texas at Brownsville, Brownsville, Texas 78520

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The expansion of the universe is often viewed as a uniform stretching of space that would affect compact objects such as atoms and stars, as well as the separation of galaxies. One usually hears that bound systems do not take part in the general expansion, but a much more subtle question is whether bound systems expand partially. In this paper, a definitive answer is given for a very simple system: a classical "atom" bound by electrical attraction. With a mathematical description appropriate for undergraduate physics majors, we show that this bound system either completely follows the cosmological expansion, or, after initial transients, completely ignores it. This all-or-nothing behavior can be understood using analysis techniques used in junior-level mechanics. We also demonstrate that this simple description is a justifiable approximation of the relativistically correct formulation of the problem.
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I. INTRODUCTION

It is not hard to explain to students that the galaxies are moving apart like pennies glued to the surface of an expanding balloon or raisins in an expanding loaf of raisin bread.[1–6] The expanding material represents the uniform stretching of space. But if space itself is stretching, does this mean that everything in it is stretching? Are galaxies growing larger? Are atoms? The usual answer is that "bound" systems do not take part in the cosmological expansion. But if space itself is stretching, how can these systems not be at least slightly affected? And what would it mean for a bound system to be "slightly affected?" Would the bound system expand at a reduced rate? That is, if the universe expands by a factor of 10^6, would a galaxy expand by, say, a factor of 10^3 ? Would less bound systems expand more closely to the full cosmological rate?

It turns out that these questions get a spectrum of different answers from experts caught unprepared. Part of the confusion is the indeterminacy of just what the question means. In this article, we will put aside some subtleties so that we can focus on a clear and simple question; in so doing, we will find a clear and interesting answer. (See Ref. 7 for the effects of cosmological expansion on clusters of galaxies; see Refs. 8 and 9 for more mathematical detail and for additional recent references.)

To try to answer our question, we consider the following simple model: a classical "atom" composed of a negative charge of negligible mass (the "electron") going around a much more massive oppositely charged "nucleus." The Coulomb binding of the atom is physically no different from the gravitational binding of a small astronomical "object," such as a solar system or a galaxy, but it allows certain technical simplifications.[10] We will place this classical atom in a homogeneous universe in which expansion is described by an expansion factor $a(t)$, where t is time. Our goal is to find the extent to which the growth of $a(t)$ causes the atom to grow, i.e., causes the electron's orbital distance to increase.

In the description of the atom, it will be useful to use two sets of spatial coordinates, both of them spherical polar coordinates with the massive nucleus at the origin. The first system consists of the *physical* coordinates (r, θ, ϕ) in which r is the proper distance from the nucleus to the electron at a given moment of time. The second is a set of *cosmological* coordinates (R, θ, ϕ); a point at fixed values of (R, θ, ϕ) is a point fixed in the stretching space of the universe that takes part in the cosmological expansion. The two coordinate systems are related by

$$r = a(t)R. \tag{1}$$

The angular coordinates θ and ϕ are the same in both the physical and the cosmological coordinates because we can think of the cosmological expansion as proceeding radially outward from the (arbitrarily chosen) origin. The question of whether the atom takes part in the cosmological expansion then boils down to whether the electron follows a trajectory of bounded r (no significant atomic expansion), approximately constant R (full cosmological expansion of the atom), or something in between?

The nature of the expansion is encoded in the functional form of $a(t)$, and the choice of this function is the choice of the kinematics of the expanding universe. The question of what does or does not expand is a kinematical question fundamentally unrelated to the physics that constrains the form of $a(t)$. For this reason, we will choose functions that lead to the clearest insights even if such choices do not correspond to perfectly realistic expansion factors.

As we shall show, the answers given by our model contain both expected and unexpected features. An expected feature is that the comparative strengths of the expansion and of the electrical binding determine whether the atom expands. An unexpected feature is that our atom undergoes expansion in an "all-or-nothing" manner. That is, a sufficiently loosely bound electron will expand with the universe and move with approximately constant R, whereas a more tightly bound electron will, after some initial disturbance of its orbit, ignore the continuing expansion and maintain a bounded r. We will see that there is no intermediate behavior. Importantly, we will find that this all-or-nothing behavior makes good physical sense.

This paper analyzes the expanding atom at two different levels. In Sec. II, our description uses only Newtonian mechanics and basic electrostatics and should be accessible to physics students in the junior year. Expansion effects are introduced in this model through a plausible heuristic stretching force in the relatively simple differential equation for the orbital radius $r(t)$. This model leads to particularly clear graphical insights in the case of cosmological expansion that is exponential in time. Numerical results for this and another model expansion are provided to reinforce the all-or-nothing feature of the atomic expansion. In Sec. III, the same classical atom is analyzed using the kinematics of general relativity and Maxwell electrodynamics in a curved spacetime.[11] The result of this analysis is a differential equation for $r(t)$ that differs only slightly from the one in Sec. II. We show, however, that the difference is not significant. If the atom is chosen to be initially nonrelativistic, then subsequent relativistic effects are unimportant. Section IV summarizes our conclusions.

II. NEWTONIAN ANALYSIS

A. Equation of motion

Our model consists of an unmoving massive nucleus fixed at the origin of a spherical polar coordinate system (r, θ, ϕ). The position of an electron of mass m orbiting in the equatorial plane $\theta = \pi/2$ is described by the functions $r(t)$ and $\phi(t)$. Because only radial forces act on the electron, its angular momentum $mr^2 d\phi/dt$ is conserved and we define the constant of motion

$$L \equiv r^2 \frac{d\phi}{dt} \tag{2}$$

to be the electron angular momentum per unit mass. In the absence of cosmological expansion effects, the equation of motion for $r(t)$ is derived in the usual way and takes the familiar form

$$\frac{d^2r}{dt^2} - \frac{L^2}{r^3} = -\frac{C}{r^2}. \tag{3}$$

In SI units, the constant of electrostatic attraction is $C = Qq/(4\pi\epsilon_0 m)$, where Qq is the magnitude of the product of the nuclear and electron charges.

Next, we need to introduce the effect of expansion. According to Eq. (1), a point fixed in the cosmological expansion—a point of constant (R, θ, ϕ)—has a radial acceleration of[12]

$$\left.\frac{d^2r}{dt^2}\right|_{\text{expansion}} = \frac{r}{a}\frac{d^2a}{dt^2}. \tag{4}$$

It seems plausible, therefore, to treat this term as a radial force per unit mass, and add it to Eq. (3) to arrive at

$$\frac{d^2r}{dt^2} - \frac{L^2}{r^3} = -\frac{C}{r^2} + \frac{r}{a}\frac{d^2a}{dt^2}. \tag{5}$$

From the solution of this equation and the chosen expansion factor $a(t)$, we can find the radial (cosmological) position $R(t)$ of the electron using Eq. (1). If we combine $r(t)$ or $R(t)$ with $\phi(t)$ from the integration of Eq. (2), we arrive at a complete description of the orbit in either physical or cosmological coordinates.

The comparative strengths of the electrostatic and cosmological terms in Eq. (5) can be usefully cast as a comparison of time scales for atomic and expansion effects. We define a characteristic atomic time scale T_{atom} as a combination of the parameters (L and C) relevant to the electron's motion

$$T_{\text{atom}} = L^3/C^2, \tag{6}$$

and note that the time for the electron to complete a circular orbit, in the absence of expansion effects, is $2\pi T_{\text{atom}}$.

B. Exponential expansion

We first choose the cosmological expansion kinematics to be exponential

$$a(t) = e^{t/T_{\exp}}, \tag{7}$$

where $T_{\exp}$ is a characteristic time for expansion. Such an expansion, a "de Sitter" cosmology,[13] is of interest in connection with inflationary models and mathematical relativity, but it is our first choice for a very different reason—such a model results in a form of Eq. (5) with no explicit time dependence:

$$\frac{d^2r}{dt^2} = \frac{L^2}{r^3} - \frac{C}{r^2} + \frac{r}{T_{\exp}^2}. \tag{8}$$

This equation is identical to the equation of motion of a particle moving in one dimension under the influence of an r-dependent potential. This view is based on the fact that Eq. (8) guarantees that the energy-like quantity

$$E \equiv \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2r^2} - \frac{C}{r} - \frac{r^2}{2T_{\exp}^2} \tag{9}$$

is constant, so the electron can be viewed as moving in an effective potential given by

$$V(r) \equiv \frac{L^2}{2r^2} - \frac{C}{r} - \frac{r^2}{2T_{\exp}^2}. \tag{10}$$

A thorough (qualitative) understanding of the motion of the electron can be obtained from a graphical analysis. A graph of the potential for various $T_{\text{atom}}/T_{\exp}$ is shown in Fig. 1, where we plot the dimensionless potential (L^2V/C^2) versus the dimensionless radial distance (Cr/L^2). Each curve is labeled with the value of the parameter $T_{\text{atom}}/T_{\exp}$ that determines how strongly the cosmological expansion affects the evolution of the atom. The larger the value of $T_{\text{atom}}/T_{\exp}$, the larger is the effect of expansion.

Expansion is absent for the top curve, for which $T_{\text{atom}}/T_{\exp} = 0$. In this case, the electron is always trapped in the potential well, i.e., it is permanently bound. If the electron begins at the bottom of the well ($r = L^2/C$ or $E = -C^2/2L^2$), it will remain in a circular orbit at that radius for all time. For any larger value of E, the electron will orbit in an ellipse. For nonzero values of $T_{\text{atom}}/T_{\exp}$, the potential at large r eventually becomes negative and decreasing, thus representing a dominant outward force. Consequently, an

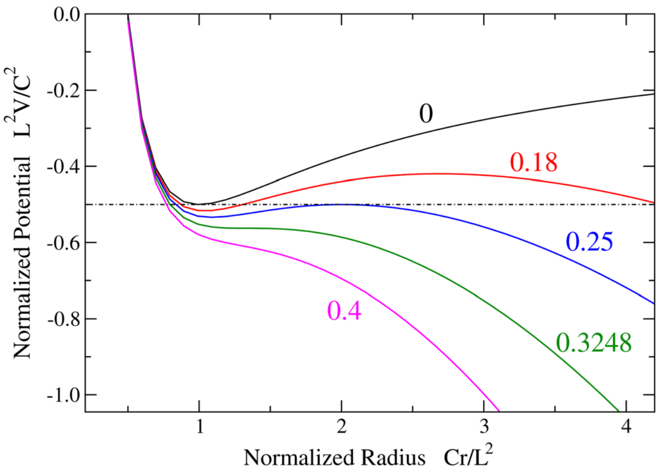

Fig. 1. Effective potential for exponential expansion. Curves are marked by the value of the parameter T_{atom}/T_{exp}. The curve labeled 0 is the no-expansion potential, for which $T_{atom}/T_{exp} = 0$. The dashed line shows the alignment of the minimum for the no-expansion potential with the local maximum of the potential for $T_{atom}/T_{exp} = 0.25$.

electron at a sufficiently large distance from the nucleus will be driven to an even larger distance. The important question is whether the electron will ever get to this region of dominant outward force. The answer is contained in the shapes of the curves in Fig. 1.

We first consider the situation where the electron sits at the bottom of the no-expansion potential well and is "surprised" when the expansion is suddenly turned on. Thus, the electron has energy $E = -C^2/2L^2$ and finds itself under the influence of one of the expansion potential curves with $T_{atom}/T_{exp} > 0$. In this scenario, there is a critical value of $T_{atom}/T_{exp} = 0.25$, above which the electron will be accelerated outward by the cosmological expansion. As shown by the dashed line in Fig. 1, this critical value occurs when the local peak in an expansion potential has the same value as the lowest point in the no-expansion well. For $0 < T_{atom}/T_{exp} < 0.25$, the electron will remain trapped in an approximately elliptical orbit.

A different scenario can also be envisioned. Imagine the electron is sitting at the bottom of an expansion potential well. In this case, the electron will remain at a fixed r (the bottom of the well), assuming such a local minimum actually exists. However, as shown in Fig. 1, there is a critical curve that separates potentials that have a local minimum from those that do not. This curve has $T_{atom}/T_{exp} = 3\sqrt{3}/16 \approx 0.3248$.

Such a qualitative analysis allows us to understand why the atom has an all-or-nothing behavior. The electron either is, or is not, trapped in the potential well. Correspondingly, the atom either expands or does not; there is no "partial expansion" possible. Underlying this graphical understanding is a broader but less precise heuristic explanation of the all-or-nothing effect, an explanation that applies regardless of the specific form of the expansion. The cosmological expansion term $r(d^2a/dt^2)/a$ increases at large physical distances r from the nucleus, whereas the centrifugal and electrical forces both decrease. This implies a sort of instability with respect to expansion. If the electron moves sufficiently far from the nucleus, the expansion term becomes more important and this pushes the electron even further away.

We can get yet another viewpoint on the bound/unbound issue by numerically solving Eq. (8). If we start the computation with the electron at the bottom of an expansion well, the results are in agreement with the predictions of the analysis based on Fig. 1—the electron remains at fixed r. More interesting is the "surprised electron" scenario discussed above (with $E = -C^2/2L^2$). The results, shown in Fig. 2, are in accord with the analysis based on Fig. 1. For T_{atom}/T_{exp} slightly greater than the 0.25 critical value, the physical radius r of the atom grows exponentially after an initial hesitation. In contrast, for T_{atom}/T_{exp} slightly less than this critical value, the electron remains trapped in an approximately elliptical orbit and is not dramatically affected by the exponential expansion.

C. Other expansion laws

It is important to check that our understanding, based on exponential expansion, applies for other expansion laws as well. For convenience, we choose an expansion law given by

$$a(t) = 1 + \left(\frac{t}{T_{exp}}\right)^2 \tanh(t/T_{exp}). \quad (11)$$

For $t/T_{exp} \gg 1$, this expansion factor is proportional to t^2, but its properties at $t = 0$ simplify our analysis. Both da/dt

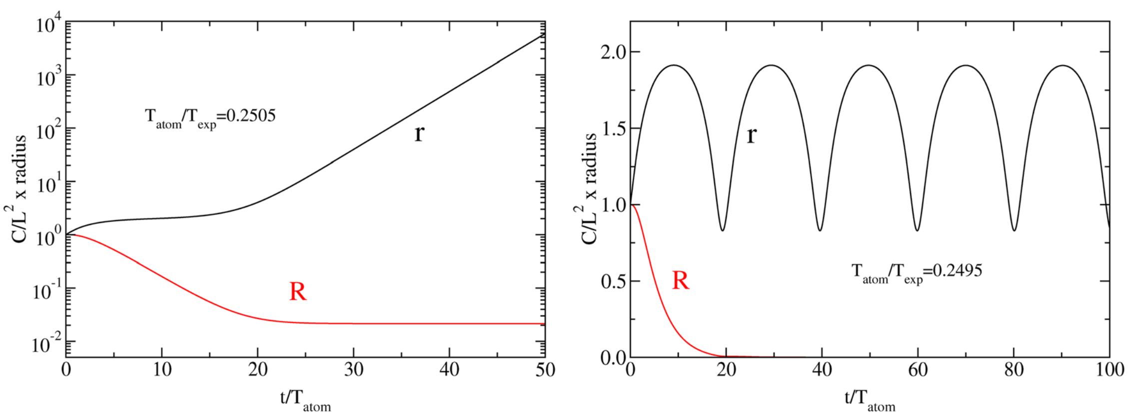

Fig. 2. Radial coordinates as a function of time for exponential expansion. On the left is the case for $T_{atom}/T_{exp} = 0.2505$ for which the electron's cosmological radius R remains approximately constant after an initial decrease to about 2% of its initial value. Due to the exponential increase in $a(t)$, the physical radius r grows without bound. On the right is the radial kinetics for a slightly smaller value of $T_{atom}/T_{exp} = 0.2495$. In this case, the electron remains bound in an approximately elliptical orbit with the physical radius oscillating between values near the original atomic radius. The coordinate radius R in this case falls off exponentially.

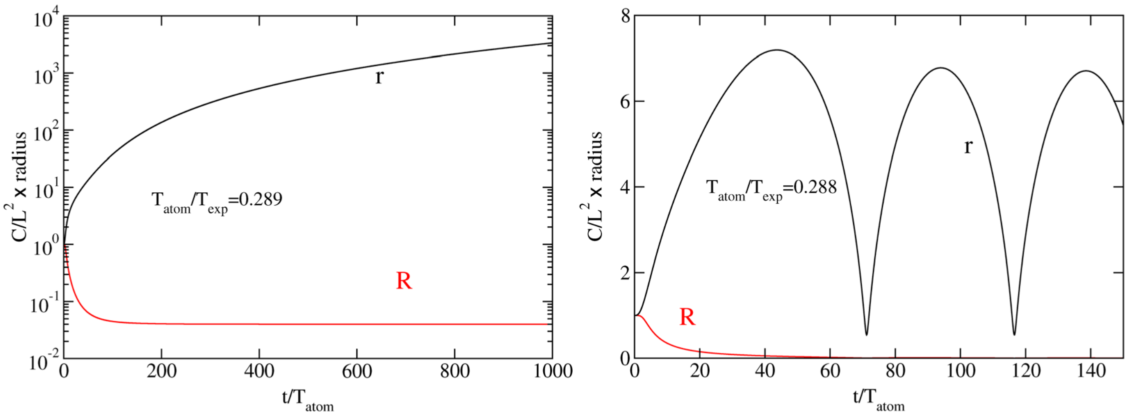

Fig. 3. Radial coordinates as a function of time for the modified $a \propto t^2$ expansion described in the text. The figure on the left is for $T_{\rm atom}/T_{\rm exp} = 0.289$, for which the electron is unbound. In this case, the cosmological radius R remains constant for large times at about 4% of its initial value, while the physical radius r expands proportional to t^2. On the right is shown a bound electron for $T_{\rm atom}/T_{\rm exp} = 0.288$. Here, the cosmological radius R decreases asymptotically to zero roughly as t^{-2}, while the physical radius oscillates as the electron orbits in a bound, approximately elliptical, orbit.

and d^2a/dt^2 vanish at $t=0$, so we can start the expansion with both $dr/dt=0$ and $dR/dt=0$. In addition, the expansion term in Eq. (5) vanishes at $t=0$, so there is no initial cosmological acceleration. Furthermore, if we choose $r=L^2/C$ to balance the Coulomb and centripetal forces, then there will be no initial acceleration.

Figure 3 shows the results of numerically solving Eq. (5) using expansion law (11) for two values of the parameter $T_{\rm atom}/T_{\rm exp}$ that are nearly the same. We see the same qualitative phenomenon as with an exponential expansion: the atom either fully takes part in the cosmological expansion and grows without bound; or, for a slightly smaller value of $T_{\rm atom}/T_{\rm exp}$, the atom does not take part in the expansion and its (physical) size remains bounded.

III. RELATIVISTIC ANALYSIS

The analysis in Sec. II is based on a heuristic term in Eq. (5) representing the effect of expansion. Here, we analyze the problem using relativistic cosmology and Maxwell-Einstein theory. We start with a standard form[14] for the spacetime metric of a homogeneous, isotropic universe

$$ds^2=-c^2dt^2+a^2(t)\left[\frac{dR^2}{1-kR^2}+R^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right]. \quad (12)$$

Here, as in Sec. II, $a(t)$ is the expansion factor and R is the cosmological radial coordinate, with $r=a(t)R$ the physical radial coordinate. As explained in more detail below, the presence of the speed of light c in the line element introduces an additional parameter for relativistic motion. For our classical atom, this parameter is the ratio of the initial orbital speed of the electron to the speed of light. The constant k in Eq. (12) can be positive, negative, or zero, and has a magnitude of order $1/R_c^2$, where R_c is a characteristic cosmological distance. If R^2/R_c^2 is not negligibly small, it means our atom occupies a significant fraction of the universe. For our purposes, we want the atom to be very small compared to the size of the universe. Thus, we omit the kR^2 term in Eq. (12) (i.e., we set $k=0$).

The first step in the relativistic analysis is to find the correct description of the electrical attraction. For the spherically symmetric electromagnetic field of the nucleus there can only be a component F^{0R} of the electromagnetic tensor $F^{\mu\nu}$. The Maxwell equations $F^{\alpha\beta}_{;\beta}=0$, with $\alpha=0$ and with $\alpha=R$, give

$$\frac{1}{R^2}\left(F^{0R}R^2\right)_{,R}=0=\frac{1}{a^3(t)}\left(F^{0R}a^3(t)\right)_{,t} \quad (13)$$

so that the solution must have the form

$$F^{0R}=\frac{Q}{R^2a^3}. \quad (14)$$

The R equation of motion of the electron's 4-velocity U^α is

$$U^\alpha U^R_{;\alpha}=\frac{q}{m}U_0F^{0R}, \quad (15)$$

where q is the magnitude of the charge of the electron. For motion in the $\theta=\pi/2$ plane this becomes, after some manipulations,

$$\frac{d}{dt}\left(a^2\frac{U^0}{c}\frac{dR}{dt}\right)-\frac{L^2}{a^2R^3(U^0/c)}=-\frac{C}{aR^2}. \quad (16)$$

Here $L\equiv U_\phi=r^2(U^0/c)d\phi/dt$ is a constant of the motion, and C is the same symbol as in Eq. (5). We now note that Eq. (5), with $r=a(t)R$, can be written in the form

$$\frac{d}{dt}\left(a^2\frac{dR}{dt}\right)-\frac{L^2}{a^2R^3}=-\frac{C}{aR^2}, \quad (17)$$

which is quite similar to Eq. (16). The adequacy of the Newtonian analysis, therefore, depends on the extent to which U^0/c differs from unity.

To compute motion for a relativistic model, Eq. (16) must be solved simultaneously with an expression for U^0/c. This additional expression is obtained by normalizing the 4-velocity $U^\mu U_\mu=-c^2$, which leads to

$$\frac{U^0}{c}=\left[1-\left(\frac{a}{c}\frac{dR}{dt}\right)^2-\left(\frac{aR}{c}\frac{d\phi}{dt}\right)^2\right]^{-1/2}. \quad (18)$$

An implicit expression for U^0/c that is useful for understanding the subsequent time-evolution is

$$\frac{U^0}{c} = \sqrt{1 + \left(\frac{a}{c}\frac{U^0}{c}\frac{dR}{dt}\right)^2 + \left(\frac{L}{aRc}\right)^2}\,. \tag{19}$$

In the Newtonian case, a model for the classical atom required only the choice of the expansion law and a value of a single dimensionless parameter $T_{\rm atom}/T_{\rm exp}$. For relativistic motion, there is an important difference: we must choose a second dimensionless parameter $\beta_0 = v_0/c$, where $v_0 = C/L$ is the initial orbital speed of the electron times 2π. This need for a second parameter is instructive. If we were to fix, say, $T_{\rm atom} = T_{\rm exp}$, then this could correspond to slow electron motion (compared to c) and slow expansion, or to fast electron motion and fast expansion. But only in the second case would relativistic effects be important.

If β_0 is not chosen small compared to unity, then relativistic effects will be important even initially. Such effects, while interesting in their own right, are not related to cosmological expansion and are beyond the scope of this paper. Rather, what is of primary interest is the question of whether an atom that is not initially relativistic can become relativistic when it is cosmologically expanding. We investigate this numerically for a universe following the exponential expansion in Eq. (7). We start with $dR/dt = 0$ and choose $\beta_0 = 0.1$ so that the electron starts out mildly relativistic. Since we want the atom to be unbounded, we take $T_{\rm atom}/T_{\rm exp}$ to be 0.252 (it turns out that with $\beta_0 \neq 0$ the unbounded behavior requires a slightly larger value of $T_{\rm atom}/T_{\rm exp}$ than in the Newtonian case).

Results for this model are shown in Fig. 4. This plot shows the unbounded growth of the physical coordinate r and also shows that $(U^0/c - 1)$, the measure of the relativistic nature of the electron, *decreases* with the expansion of the atom. The mathematical basis for this decrease is not hard to understand from Eq. (19). At large expansion, Eq. (16) tells us that the combination $a^2(U^0/c)dR/dt$ is approximately constant. This means that the middle term inside the square root of Eq. (19) must fall off as a^{-2}. The last term in the square root also falls off with the expansion. The implication, validated by Fig. 4, is that $(U^0/c - 1) \to 0$ with unbounded expansion.

The mathematical "how" is then clear, but the physical "why" must be explained. To this end, it is interesting to consider the velocity of an unbound electron relative to the "fabric of the universe,"—the velocity $v_{\rm loc}$ that would be measured in the local Minkowski frame of an observer comoving with the cosmological expansion,[15] i.e., an observer with constant (R, θ, ϕ). This velocity can be shown to be

$$v_{\rm loc} = c\sqrt{1 - (c/U^0)^2}\,, \tag{20}$$

demonstrating that as the expansion proceeds, the particle is, in some sense, becoming less relativistic.

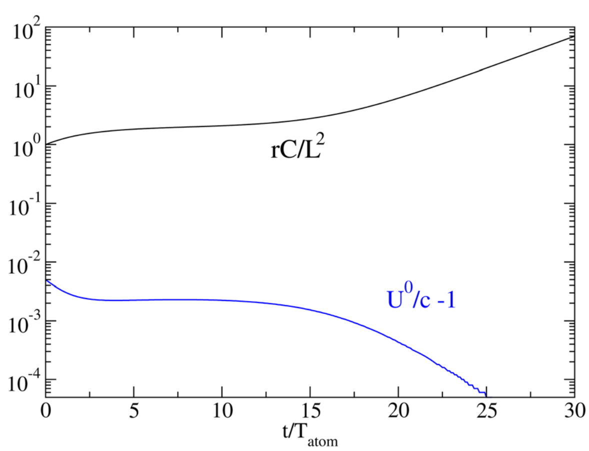

Fig. 4. The unbound physical radius and the index of relativistic effects $U^0/c - 1$ for exponential expansion with $T_{\rm atom}/T_{\rm exp} = 0.252$, starting with $dR/dt = 0$ and $\beta_0 = 0.1$.

IV. CONCLUSION

In this paper, we have presented a simple definitive question about the influence of the expansion of the universe on a very particular physical system: a classical "atom." Our analysis provides a simple definitive answer: expansion forces increase with increasing atomic radius, while atomic forces decrease. This amounts to an instability with respect to the disruption of an atom. If the atomic accelerations are initially larger than the cosmological accelerations, then the subsequent expansion will become less and less important. The atom will not "partially" take part in the expansion. If, on the other hand, the cosmological effect is initially stronger, the atomic radius will increase and the atomic forces will become less and less important, and the atom will fully take part in the expansion.

In analyzing this problem, we have relied on a simple description of expansion, described in Eq. (5), which avoids relativistic effects. A major pedagogical point is the simple way in which the "what expands" question can be graphically understood for the special case of exponential cosmological expansion.

Using a fully general relativistic calculation, we have shown that the simple nonrelativistic model is fully adequate. We have also shown that for an atom that expands with the universe, relativistic effects become *less* important as the atom gets larger.

We end with a practical consideration. Our quantification of the relative strengths of atomic and expansion forces is given in terms of a characteristic time $T_{\rm atom}$ for the motion of electrons in atoms, and a cosmological expansion time $T_{\rm exp}$ (e.g., the Hubble time). Our analyses show that atomic forces are initially stronger if $T_{\rm atom}/T_{\rm exp}$ is less than order unity. Because $T_{\rm atom} \sim 10^{-16}$ s and $T_{\rm exp} \sim 4 \times 10^{17}$ s, we see that atoms are in no danger of being disrupted by cosmological expansion.

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[13]See, e.g., Sec. 27.11 of Ref. 6.
[14]Ref. 6, Chap. 27.
[15]This viewpoint has been stressed by E. F. Bunn and D. W. Hogg, "The kinematic origin of the cosmological redshift," Am. J. Phys. **77**, 688–694 (2009), and is closely related to the analysis in Ref. 9.