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Republication of: A homogeneous universe of constant mass and increasing radius accounting for the radial velocity of extra-galactic nebulae

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A HOMOGENEOUS UNIVERSE OF CONSTANT MASS AND INCREASING RADIUS ACCOUNTING FOR THE RADIAL VELOCITY OF EXTRAGALACTIC NEBULAE

Note by Abbé G. Lemaître

1. Generalities.

According to the theory of relativity, a homogeneous universe may exist not only when the distribution of matter is uniform, but also when all positions in space are completely equivalent, there is no center of gravity. The radius R of space is constant, space is elliptic with uniform positive curvature $1/R^2$, the lines starting from a same point come back to their starting point after having travelled a path equal to πR, the total volume of space is finite and equal to $\pi^2 R^3$, straight lines are closed lines going through the whole space without encountering any boundary (1).

Two solutions have been proposed. That of de Sitter ignores the existence of matter and supposes its density equal to zero. It leads to special difficulties of interpretation which we will be referred to later, but it is of great interest as explaining the fact that extragalactic nebulæ seem to recede from us with a huge velocity, as a simple consequence of the properties of the gravitational field, without having to suppose that we are at a point of the universe distinguished by special properties.

The other solution is that of Einstein. It pays attention to the obvious fact that the density of matter is not zero and it leads to a relation between this density and the radius of the universe. This relation forecasted the existence of masses enormously greater than any known when the theory was for the first time compared with the facts. These masses have since been discovered, the distances and dimensions of extragalactic nebulæ having become established. From Einstein's formula and recent observational data, the radius of the universe is found to be some hundred times greater than the most distant objects which can be photographed by our telescopes (2).

Each theory has its own advantage. One is in agreement with the observed radial velocities of nebulæ, the other with the existence of matter, giving a satisfactory relation between the radius and the mass of the universe. It

(1) We consider simply connected elliptic space, i.e. without antipodes.

(2) Cf. Hubble E. Extra-Galactic Nebulæ, *ApJ.*, vol. 64, p. 321, 1926. *Mt Wilson Contr.* N° 324.

seems desirable to find an intermediate solution which could combine the advantages of both.

At first sight, such an intermediate solution does not appear to exist. A static gravitational field with spherical symmetry has only two solutions, that of Einstein and that of de Sitter, if the matter is uniformly distributed without pressure or internal stress. De Sitter's universe is empty, that of Einstein has been described as containing as much matter as it can contain. It is remarkable that the theory can provide no mean between these two extremes.

The solution of the paradox is that the de Sitter's solution does not really meet all the requirements of the problem ([3]). Space is homogeneous with constant positive curvature; space-time is also homogeneous, for all events are perfectly equivalent. But the partition of space-time into space and time disturbs the homogeneity. The selected coordinates introduce a center to which nothing corresponds in reality; a particle at rest somewhere else than at the center does not describe a geodesic. The coordinates chosen destroy the homogeneity that exists in the data for the problem and produce the paradoxical results which appear at the so-called «horizon» of the center. When we use coordinates and a corresponding partition of space and time of such a kind as to preserve the homogeneity of the universe, the field is

([3]) Cf. K. Lanczos, Bemerkung zur de Sitterschen Welt, *Phys. Zeitschr.* vol. 23, 1922, p. 539, and H. Weyl, Zur allgemeinen Relativitätstheorie, Id., vol. 24, 1923, p. 230, 1923. We follow the point of view of Lanczos here. The worldlines of nebulæ form a bunch with ideal center and real axial hyperplane; space orthogonal to these worldlines is formed by the hyperspheres equidistant from the axial plane. This space is elliptic, its variable radius being minimum at the moment corresponding to the axial plane. Following the assumption of Weyl, the worldlines are parallel in the past; the normal hypersurfaces representing space are horospheres, the geometry of space is thus Euclidean. The spatial distance between nebulæ increases as the parallel geodesics which they follow recede one from the other proportionally to $e^{t/R}$, where t is the proper time and R the radius of the universe. The Doppler effect is equal to r/R, where r is the distance from the source at the moment of observation. Cf. G. Lemaître, Note on de Sitter's universe, *Journal of mathematics and physics*, vol. 4, n°3, May 1925, or *Publications du Laboratoire d'Astronomie et de Géodesie de l'Université de Louvain*, vol. 2, p. 37, 1925. For the discussion of the de Sitter's partition, see P. Du Val, Geometrical note on de Sitter's world, *Phil. Mag.* (6), vol. 47, p. 930, 1924. Space is constituted by hyperplanes orthogonal to a time line described by the introduced center, the trajectories of nebulæ are the trajectories orthogonal to these planes, in general they are no more geodesics and they tend to becoming lines of null length when one approaches the horizon of the center, i.e. the polar hyperplane of the central axis with respect to the absolute one.

found to be no longer static ; the universe becomes of the same form as that of Einstein, with a radius of space no longer constant but varying with the time according to a particular law ([4]).

In order to find a solution combining the advantages of those of Einstein and de Sitter, we are led to consider an Einstein universe where the radius of space (or of the universe) is allowed to vary in an arbitrary way.

2. Einstein universe of variable radius. Field equations. Conservation of energy.

As in Einstein's solution, we liken the universe to a rarefied gas whose molecules are the extragalactic nebulæ . We suppose them so numerous that a volume small in comparison with the universe as a whole contains enough nebulæ to allow us to speak of the density of matter. We ignore the possible influence of local condensations. Furthermore, we suppose that the nebulæ are uniformly distributed so that the density does not depend on position.

When the radius of the universe varies in an arbitrary way, the density, uniform in space, varies with time. Furthermore, there are generally internal stresses which, in order to preserve the homogeneity, must reduce to a simple pressure, uniform in space and variable with time. The pressure, being two-thirds of the kinetic energy of the molecules, is negligible with respect to the energy associated with matter; the same can be said of interior stresses in nebulæ or in stars belonging to them. We are thus led to put $p = 0$. Nevertheless it might be necessary to take into account the radiation-pressure of electromagnetic energy travelling through space; this energy is weak but it is evenly distributed through the whole of space and might provide a notable contribution to the mean energy. We shall keep the pressure p in the general equations as the average radiation-pressure of light, but we shall write $p = 0$ when we discuss the application to astronomy.

([4]) If we restrict the problem to two dimensions, one of space and one of time, the partition of space and time used by Sitter can be represented on a sphere: the lines of space are provided by a system of great circles which intersect on a same diameter, and the lines of time are the parallels cutting orthogonally the lines of space. One of these parallels is a great circle and thus a geodesic, it corresponds to the center of space, the pole of this great circle is a singular point corresponding to the horizon of the center. Of course the representation must be extended to four dimensions and the time coordinate must be assumed imaginary, but the defect of homogeneity resulting from the choice of the coordinates remains. The coordinates respecting the homogeneity require taking a system of meridian lines as lines of time and the corresponding parallels for lines of space, whereas the radius of space varies with time.

We denote the density of total energy by ρ, the density of radiation energy by $3p$, and the density of the energy condensed in matter by $\delta = \rho - 3p$.

We identify ρ and $-p$ with the components T_4^4 and $T_1^1 = T_2^2 = T_3^3$ of the material energy tensor, and δ with T. Working out the contracted Riemann tensor for a universe with a line-element given by

$$ds^2 = -R^2 d\sigma^2 + dt^2 \tag{1}$$

where $d\sigma$ is the elementary distance in a space of radius unity, and the radius of space R is a function of time, we find that the field equations can be written

$$3\frac{R'^2}{R^2} + \frac{3}{R^2} = \lambda + \kappa\rho \tag{2}$$

and

$$2\frac{R''}{R} + \frac{R'^2}{R^2} + \frac{1}{R^2} = \lambda - \kappa\rho \tag{3}$$

Accents denote derivatives with respect to t; λ is the cosmological constant whose value is unknown, and κ is the Einstein constant whose value is $1,87 \times 10^{-27}$ in C.G.S. units (8π in natural units).

The four identities expressing the conservation of momentum and of energy reduce to

$$\frac{d\rho}{dt} + \frac{3R'}{R}(\rho + p) = 0 \tag{4}$$

which is the conservation of energy equation. This equation can replace (3). It is suitable for an interesting interpretation. Introducing the volume of space $V = \pi^2 R^3$, it can be written

$$d(V\rho) + pdV = 0 \tag{5}$$

showing that *the variation of total energy plus the work done by radiation-pressure is equal to zero.*

3. Case of a universe of constant total mass.

Let us seek a solution for which the total mass $M = V\delta$ remains constant. We can write

$$\kappa\delta = \frac{\alpha}{R^3} \tag{6}$$

where α is a constant. Taking account of the relation

$$\rho = \delta + 3p$$

existing between the various kinds of energy, the principle of conservation of energy becomes

$$3d(pR^3) + 3pR^2 dR = 0 \tag{7}$$

whose integration is immediate; and, β being a constant of integration,

$$\kappa p = \frac{\beta}{R^4} \tag{8}$$

and therefore

$$\kappa \rho = \frac{\alpha}{R^3} + \frac{3\beta}{R^4} \tag{9}$$

By substitution in (2) we have to integrate

$$\frac{R'^2}{R^2} = \frac{\lambda}{3} - \frac{1}{R^2} + \frac{\kappa\rho}{3} = \frac{\lambda}{3} - \frac{1}{R^2} + \frac{\alpha}{3R^3} + \frac{\beta}{R^4} \tag{10}$$

or

$$t = \int \frac{dR}{\sqrt{\frac{\lambda R^2}{3} - 1 + \frac{\alpha}{3R} + \frac{\beta}{R^2}}} \tag{11}$$

When α and β vanish, we obtain the de Sitter solution ([5])

$$R = \sqrt{\frac{3}{\lambda}} \cosh \sqrt{\frac{\lambda}{3}} \, (t - t_0) \tag{12}$$

The Einstein solution is found by making $\beta = 0$ and R constant. Writing $R' = R'' = 0$ in (2) and (3) we find

$$\frac{1}{R^2} = \lambda \qquad \frac{3}{R^2} = \lambda + \kappa\rho \qquad \rho = \delta$$

or

$$R = \frac{1}{\sqrt{\lambda}} \qquad \kappa\rho = \frac{2}{R^2} \tag{13}$$

and from (6)

$$\alpha = \kappa\delta R^3 = \frac{2}{\sqrt{\lambda}} \tag{14}$$

The Einstein solution does not result from (14) alone, it also supposes that the initial value of R' is zero. Indeed, if, in order to simplify the notation, we write

$$\lambda = \frac{1}{R_0^2} \tag{15}$$

([5]) Cf. Lanczos, *l.c.*

and put in (11) $\beta = 0$ and $\alpha = 2R_0$, it follows that

$$t = R_0\sqrt{3}\int \frac{dR}{R - R_0}\sqrt{\frac{R}{R + 2R_0}} \tag{16}$$

For this solution the two equations (13) are of course no longer valid. Writing

$$\kappa\delta = \frac{2}{R_E^2} \tag{17}$$

we have from (14) and (15)

$$R^3 = R_E^2 R_0 \tag{18}$$

The value of R_E, the radius of the universe computed from the average density by Einstein's equations (17), has been found by Hubble to be

$$R_E = 8,5 \times 10^{28}\text{cm.} = 2,7 \times 10^{10}\text{parsecs} \tag{19}$$

We shall see later that the value of R_0 can be computed from the radial velocities of the nebulæ; R can then be found from (18). Finally, we shall show that a solution introducing a relation substantially different from (14) would lead to consequences not easily acceptable.

4. Doppler effect due to the variation of the radius of the universe

From (1) giving the line element of the universe, the equation for a light ray is

$$\sigma_2 - \sigma_1 = \int_{t_1}^{t_2} \frac{dt}{R} \tag{20}$$

where σ_1 and σ_2 relate to spatial coordinates. We suppose that the light is emitted at the point σ_1 and observed at σ_2.

A ray of light emitted slightly later starts from σ_1 at time $t_1 + \delta t_1$ and reaches σ_2 at time $t_2 + \delta t_2$. We have therefore

$$\frac{\delta t_2}{R_2} - \frac{\delta t_1}{R_1} = 0, \qquad \frac{\delta t_2}{\delta t_1} - 1 = \frac{R_2}{R_1} - 1 \tag{21}$$

where R_1 and R_2 are the values of the radius R at the time of emission t_1 and at the time of observation t_2. t is the proper time; if δt_1 is the period of the emitted light, t_2 is the period of the observed light. Moreover, δt_1 can also be considered as the period of the light emitted under the same conditions in the neighbourhood of the observer, because the period of the light emitted under the same physical conditions has the same value everywhere when reckoned in proper time. Therefore

$$\frac{v}{c} = \frac{\delta t_2}{\delta t_1} - 1 = \frac{R_2}{R_1} - 1 \tag{22}$$

measures the apparent Doppler effect due to the variation of the radius of the universe. *It equals the ratio of the radii of the universe at the instants of observation and emission, diminished by unity.* v is that velocity of the observer which would produce the same effect. When the source is near enough, we can write approximately

$$\frac{v}{c} = \frac{R_2 - R_1}{R_1} = \frac{dR}{R} = \frac{R'}{R}dt = \frac{R'}{R}r$$

where r is the distance of the source. We have therefore

$$\frac{R'}{R} = \frac{v}{cr} \tag{23}$$

Radial velocities of 43 extragalactic nebulæ are given by Strömberg ([6]).

The apparent magnitude m of these nebulæ can be found in the work of Hubble. It is possible to deduce their distance from it, because Hubble has shown that extragalactic nebulæ have approximately equal absolute magnitudes (magnitude $= -15.2$ at 10 parsecs, with individual variations ± 2), the distance r expressed in parsecs is then given by the formula $\log r = 0,2m + 4,04$.

One finds a distance of about 10^6 parsecs, varying from a few tenths to 3,3 megaparsecs. The probable error resulting from the dispersion of absolute magnitudes is considerable. For a difference in absolute magnitude of ± 2, the distance exceeds from 0,4 to 2,5 times the calculated distance. Moreover, the error is proportional to the distance. One can admit that, for a distance of one megaparsec, the error resulting from the dispersion of magnitudes is of the same order as that resulting from the dispersion of velocities. Indeed, a difference of magnitude of value unity corresponds to a proper velocity of 300 Km/s, equal to the proper velocity of the sun compared to nebulæ . One can hope to avoid a systematic error by giving to the observations a weight proportional to $\frac{1}{\sqrt{1+r^2}}$, where r is the distance in megaparsecs.

([6]) Analysis of radial velocities of globular clusters and non galactic nebulæ. *Ap.J.* vol. 61, p. 353, 1925. *M^t Wilson Contr.*, N° 292.

Using the 42 nebulæ appearing in the lists of Hubble and Strömberg (7), and taking account of the proper velocity of the Sun (300 Km/s in the direction $\alpha = 315°$, $\delta = 62°$), one finds a mean distance of 0,95 megaparsecs and a radial velocity of 600 Km/sec, i.e. 625 Km/sec at 10^6 parsecs (8).

We will thus adopt

$$\frac{R'}{R} = \frac{v}{rc} = \frac{625 \times 10^5}{10^6 \times 3,08 \times 10^{18} \times 3 \times 10^{10}} = 0,68 \times 10^{-27}\mathrm{cm}^{-1} \tag{24}$$

This relation enables us to calculate R_0. We have indeed by (16)

$$\frac{R'}{R} = \frac{1}{R_0\sqrt{3}}\sqrt{1 - 3y^2 + 2y^3} \tag{25}$$

where we have set

$$y = \frac{R_0}{R} \tag{26}$$

On the other hand, from (18) and (26)

$$R_0^2 = R_E^2 y^3 \tag{27}$$

and therefore

$$3\left(\frac{R'}{R}\right)^2 R_E^2 = \frac{1 - 3y^2 + 2y^3}{y^3} \tag{28}$$

With the adopted numerical data (24) for $\frac{R'}{R}$ and (19) for R_E, we have

$$y = 0,0465.$$

(7) Account is not taken of N.G.C. 5194 which is associated with N.G.C. 5195. The introduction of the Magellanic clouds would be without influence on the result.

(8) By not giving a weight to the observations, one would find 670 Km/sec at 1.16×10^6 parsecs, 575 Km/sec at 10^6 parsecs. Some authors sought to highlight the relation between v and r and obtained only a very weak correlation between these two terms. The error in the determination of the individual distances is of the same order of magnitude as the interval covered by the observations and the proper velocity of nebulæ (in any direction) is large (300 Km/sec according to Strömberg), it thus seems that these negative results are neither for nor against the relativistic interpretation of the Doppler effect. The inaccuracy of the observations makes only possible to assume v proportional to r and to try to avoid a systematic error in the determination of the ratio v/r. Cf. Lundmark, The determination of the curvature of space time in de Sitter's world, M.N., vol. 84, p. 747, 1924, and Strömberg, *l.c.*

We have therefore

$$R = R_E\sqrt{y} = 0,215 R_E = 1,83 \times 10^{28}\text{cm.} = 6 \times 10^9 \text{ parsecs}$$

$$R_0 = Ry = R_E y^{\frac{3}{2}} = 8,5 \times 10^{26}\text{cm.} = 2,7 \times 10^8 \text{ parsecs}$$
$$= 9 \times 10^8 \text{ light years.}$$

Integral (16) can easily be computed. Writing

$$x^2 = \frac{R}{R + 2R_0} \tag{29}$$

it can be written

$$t = R_0\sqrt{3}\int \frac{4x^2 dx}{(1-x^2)(3x^2-1)} = R_0\sqrt{3}\log\frac{1+x}{1-x} + R_0\log\frac{\sqrt{3}x-1}{\sqrt{3}x+1} + C \tag{30}$$

If σ is the fraction of the radius of the universe travelled by light during time t, we have also from (20)

$$\sigma = \int \frac{dt}{R} = \sqrt{3}\int\frac{2dx}{3x^2-1} = \log\frac{\sqrt{3}x-1}{\sqrt{3}x+1} + C'. \tag{31}$$

The following table gives values of σ and t for different values of $\frac{R}{R_0}$.

$\frac{R}{R_0}$	$\frac{t}{R_0}$	σ RADIANS	σ DEGREES	$\frac{v}{c}$
1	$-\infty$	$-\infty$	$-\infty$	19
2	$-4,31$	$-0,889$	$-51°$	9
3	$-3,42$	$-0,521$	$-30°$	$5\frac{2}{3}$
4	$-2,86$	$-0,359$	$-21°$	4
5	$-2,45$	$-0,266$	$-15°$	3
10	$-1,21$	$-0,087$	$-\ 5°$	1
15	$-0,50$	$-0,029$	$-\ 1°7$	$\frac{1}{3}$
20	0	0	0	0
25	$0,39$	$0,017$	$1°$	
∞	∞	$0,087$	$5°$	

The constants of integration are adjusted to make σ and t vanish for $\frac{R}{R_0} = 20$ in place of 21,5. The last column gives the Doppler effect computed from (22). The approximate formula (23) would make $\frac{v}{c}$ proportional to r and thus to σ. The error committed by adopting this equation is only 0.005 for $\frac{v}{c} = 1$. The approximate formula may therefore be used within the limits of the visible spectrum.

5. The meaning of equation (14).

The relation (14) between the two constants λ and σ has been adopted following Einstein's solution. It is the necessary condition that quartic under the radical in (11) may have a double root R_0 giving on integration a logarithmic term. For simple roots, integration would give a square root, corresponding to a minimum of R as in de Sitter's solution (12). This minimum would generally occur at time of the order of R_0, say 10^9 years, i.e. quite recently for stellar evolution. It thus seems that the relation existing between the constants α and λ must be close to (14) for which this minimum is removed to the epoch at minus infinity ([9]).

6. Conclusion.

We have found a solution such that:

1. The mass of the universe is a constant related to the cosmological constant by Einstein's relation

$$\sqrt{\lambda} = \frac{2\pi^2}{\kappa M} = \frac{1}{R_0}$$

2. The radius of the universe increases without limits from an asymptotic value R_0 for $t = -\infty$.

3. The recession velocities of extragalactic nebulæ are a cosmical effect of the expansion of the universe. The initial radius R_0 can be computed by formulæ (24) and (25) or by the approximate formula $R_0 = \frac{rc}{v\sqrt{3}}$.

4. The radius of the universe is of the same order of magnitude as the radius R_E deduced from density according to Einstein's formula

$$R = R_E \sqrt[3]{\frac{R_0}{R_E}} = \frac{1}{5} R_E$$

([9]) If the positive roots were to become imaginary, the radius would vary from zero upwards, the variation slowing down in the neighbourhood of the modulus of the imaginary roots. For a relation substantially different from (14), this slowing down becomes weak and the time of evolution after leaving $R = 0$ becomes again of the order of R_0.

This solution combines the advantages of the Einstein and de Sitter solutions.

Note that the largest part of the universe is forever out of our reach. The range of the 100-inch Mount Wilson telescope is estimated by Hubble to be 5×10^7 parsecs, or about $\frac{1}{120}R$. The corresponding Doppler effect is 3000 Km/sec. For a distance of $0,087R$ it is equal to unity, and the whole visible spectrum is displaced into the infra-red. It is impossible that ghost images of nebulæ or suns would form, as even if there were no absorption these images would be displaced by several octaves into the infra-red and would not be observed.

It remains to find the cause of the expansion of the universe. We have seen that the pressure of radiation does work during the expansion. This seems to suggest that the expansion has been set up by the radiation itself. In a static universe, light emitted by matter travels round space, comes back to its starting point and accumulates indefinitely. It seems that this may be the origin of the velocity of expansion R'/R which Einstein assumed to be zero and which in our interpretation is observed as the radial velocity of extragalactic nebulæ.