Prof. Ted Jacobson
Room 3151 PSC, (301)405-6020
jacobson@umd.edu

Relativistic Beaming

Consider a source of radiation that emits isotropically in its own rest frame $\mathcal{S}_*$. If the source is moving with velocity v in the x-direction of an inertial frame $\mathcal{S}$, the flux will not be isotropic in $\mathcal{S}$ but will rather be concentrated towards the forward direction. This is called relativistic beaming and is very important in high energy astrophysics.

1. (a) A photon with frequency ω_* travels with angle θ_* from the x-direction in the frame $\mathcal{S}_*$. Find the frequency ω and angle θ of travel from the x-axis in the frame $\mathcal{S}$. Show that the angle is given by

$$\cos\theta = \frac{k_x}{|\mathbf{k}|} = \frac{\cos\theta_* + v}{1 + v\cos\theta_*} \tag{1}$$

or (which is simpler for taking the small angle limit)

$$\tan\theta = \frac{k_y}{k_x} = \frac{\sin\theta_*}{\gamma(\cos\theta_* + v)}, \tag{2}$$

where $\mathbf{k}$ is the photon wavevector, and we use units with $c = 1$. (Note that one can find the inverse relations by interchanging θ and θ_* and replacing v by $-v$.)

   (b) To what angle θ does $\theta_* = \pi/2$ correspond? What angle θ_* corresponds to $\theta = \pi/2$?

2. Suppose two photons are emitted at angles θ and $\theta + \delta\theta$ from the moving source, with a time separation Δt_e, and suppose both photons reach a distant observer at rest in the frame $\mathcal{S}$. (Since the observer is distant the angle difference $\delta\theta$ can be neglected.) Show that the time separation of observation of the two photons is given by

$$\Delta t_o = (1 - v\cos\theta)\Delta t_e, \tag{3}$$

where both times are measured in the frame $\mathcal{S}$.

3. (a) The *specific intensity* I_ω at frequency ω is defined by $I_\omega = dE/d\omega dt d\Omega$, where dE is the energy in the frequency range $d\omega$ passing in a time dt through a surface subtending a solid angle $d\Omega$. Show that the ratio of specific intensities seen in the two frames is

$$I_\omega/I_{\omega_*} = (\omega/\omega_*)^3 = \Big(\gamma(1 - v\cos\theta)\Big)^{-3} \tag{4}$$

where γ is the usual relativistic gamma factor $(1 - v^2)^{-1/2}$. [*Hint*: Compare the radiation energy that emerges between the angles θ_* and $\theta_* + d\theta_*$ during a time dt_* in the frame $\mathcal{S}_*$ with the corresponding energy received by the observer in the frame $\mathcal{S}$.]

   (b) Show that the forward intensity ratio is given by

$$I_\omega(0)/I_{\omega_*} = \gamma^3(1 + v)^3. \tag{5}$$

In the limit where v is very close to the speed of light, this memorably becomes $8\gamma^3$. Note that for $\gamma = 10$ this is already of order 10^4! Sources beamed toward the viewer can appear *much* brighter than in their rest frame.