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Vacuum

The field operators $\phi(x)$ and $\phi(y)$ commute when x and y are spacelike separated. The same is true for operators locally constructed out of the field operators. So we get this notion of the local algebra of operators associated with a region. In algebraic QFT, this is part of the axiomatic framework.

Reeh-Schlieder theorem: Consider any region R in spacetime, and the algebra of operators $\mathcal{A}_\mathcal{R}$ localized in R, meaning the operators that commute with every operator in the "causal complement" of R. The set of states of the form $A|0\rangle$, with $A \in \mathcal{A}_R$ is dense in the full Hilbert space.

A little analogy for this theorem is provided by the spin singlet state in the Hilbert space of two spin-1/2 degrees of freedom. The Hilbert space is the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, which is four dimensional. The singlet state is $|0\rangle = (|+-\rangle - |-+\rangle)/\sqrt{2}$. All of the states in the Hilbert space can be generated by acting on $|0\rangle$ with operators that act only on the first spin, $\mathcal{H}_1$:

$$S_1^z|0\rangle = (|+-\rangle + |-+\rangle)/2\sqrt{2} \tag{1}$$

$$S_1^+|0\rangle = -|++\rangle/\sqrt{2} \tag{2}$$

$$S_1^-|0\rangle = |--\rangle/\sqrt{2} \tag{3}$$

This is possible thanks to the entanglement of the singlet state, and the same is true for quantum fields.

The vacuum state of a quantum field is entangled because the $(\vec{\nabla}\phi)^2$ term in the Hamiltonian couples the field values at neighboring points. For a free field, the ground state is the ground state of the normal modes, which is highly entangled with respect to degrees of freedom localized in space. A sign of this entanglement is the fact that the correlation function $\langle 0|\phi(\vec{x},t)\phi(\vec{y},t)|0\rangle$ is nonzero, while $\langle 0|\phi(\vec{x},t)|0\rangle = 0$.

The Hilbert space of a quantum field theory can be thought of roughly as a tensor product of Hilbert spaces at each point, $\otimes_{\vec{x}}\mathcal{H}_{\vec{x}}$. For instance, suppose we make a lattice approximation to space, with a discrete set of points labeled by an index i. Then the field takes a value at each lattice point, ϕ_i, and the quantum state is a function of all these values, $\Psi(\phi_1, \phi_2, \dots)$. The space of such functions is the tensor product of the Hilbert spaces associated with the lattice points, $\otimes_i\mathcal{H}_i$.

Suppose we divide space at one time into two regions L and R, separated by the yz plane. We can think of the total Hilbert space of a quantum field roughly as the tensor product $\mathcal{H}_L \otimes \mathcal{H}_R$. Let $\mathcal{O}_R$ be any observable localized in the causal domain determined by the right half space $x > 0$ at $t = 0$ — a.k.a. the *Rindler wedge*. Then its vacuum expectation value is given by the density matrix arising from the partial trace of the vacuum projection operator $|0\rangle\langle 0|$ over the left factor $\mathcal{H}_L$ of the Hilbert space:

$$\langle 0|\mathcal{O}_R|0\rangle = \mathrm{Tr}(\rho_R\mathcal{O}), \quad \text{where} \quad \rho_R = \mathrm{Tr}_L|0\rangle\langle 0|. \tag{4}$$

It shouldn't surprise you that ρ_R is not a pure state; rather, it is a mixed state, because of the spatial entanglement in the vacuum. What is really remarkable, however, is that, for any relativistic quantum field theory, ρ_R has precisely the form of a *thermal* state, with respect to the Hamiltonian H_B that generates Lorentz boosts.

To understand the nature of Lorentz boosts in a simple way just consider the Minkowski metric written in (hyperbolic) polar coordinates:

$$ds^2 = dt^2 - dx^2 = l^2 d\eta^2 - dl^2. \tag{5}$$

The coordinate relation is

$$x = l \cosh \eta, \quad t = l \sinh \eta, \tag{6}$$

in perfect analogy with polar coordinates on the Euclidean plane, and the coordinate η is the hyperbolic angle. The Lorentz boost symmetry is simply η translation, which moves points along the timelike hyperbolae at fixed l.

Now just as the angular momentum operator J_z generates ordinary rotations in the xy plane, there is a boost generator H_B that generates hyperbolic rotations in the tx Minkowski space plane. In terms of this "boost Hamiltonian", the vacuum density matrix takes the form of a thermal, Gibbs state,

$$\rho_R = Z^{-1} e^{-H_B/T_U}, \quad \text{where} \quad T_U = \hbar/2\pi \text{ is the } \textit{Unruh tempreature.} \tag{7}$$

How does this temperature with dimensions of action relate to "real" temperature? Well an observer on a hyperbolic trajectory at a fixed l has proper time interval $d\tau = l d\eta$. And l is actually the inverse of the acceleration (like the radius is the inverse of the curvature of a circle), $l = a^{-1}$. So we also have $d\eta = a d\tau$. So if we want to scale H_B to the generator of proper time translations on a given hyperbola, we need only multiply it by a: $H_B d\eta = (H_B a) d\tau$. Then we must correspondingly multiply the temperature T_U by a, which yields the Unruh temperature for an accelerating observer, $T = \hbar a/2\pi$.

Being that it is a mixed state, ρ_R has nonzero von Neumann entropy, $S = -\mathrm{Tr}\rho_R \ln \rho_R$. This entropy is infinite. A quick way to see why — and how — is to think of the vacuum as a thermal bath with a local temperature $T = \hbar/2\pi l$, which diverges as the yz plane at $l = 0$ is approached. In four spacetime dimensions, the entropy density of a massless field in thermal equilibrium is proportional to $T^3/\hbar^3$ — this follows from scale invariance: the entropy density has dimensions of inverse length cubed, and the only quantity with dimensions of energy is the temperature. Integrating the entropy density thus yields (note that dl measures proper length at fixed η)

$$S \sim \int dA \int_\epsilon^\infty dl\, l^{-3} \sim A/\epsilon^2, \tag{8}$$

where A is the (infinite) area of the yz plane, and ϵ is a short distance cutoff just to regulate the divergence coming from the infinite acceleration temperature. Note that almost all the entropy comes from degrees of freedom close to the yz plane, and the entropy therefore scales as the area of the plane. If there were some reason to cut off the integral, the entropy per unit area would be finite. If the cutoff is placed at the Planck length $l_P = \sqrt{\hbar G/c^3}$, the result is of the order of the Bekenstein-Hawking black hole entropy, $A/4l_P^2$.

Finally, it is not hard to be more specific about the nature of the entanglement between $\mathcal{H}_L$ and $\mathcal{H}_R$ in the vacuum state. Indeed, the vacuum state has the form

$$|0\rangle \propto \sum_n e^{-\frac{\pi}{\hbar} E_n} |\bar{n}\rangle |n\rangle, \tag{9}$$

where n labels a basis of eigenstates of the boost hamiltonian, E_n is their boost energy eigenvalue, and $|\bar{n}\rangle \in \mathcal{H}_L$ is the CPT conjugate to the state $|n\rangle \in \mathcal{H}_R$. The reduced density matrix formed by the partial trace over $\mathcal{H}_L$ yields (7).