Index Notation

Contravariant and covariant vectors

Under a coordinate transformation $x^\mu \to y^\mu(x^\alpha)$, the coordinate differentials transform as

$$dy^\mu = \frac{\partial y^\mu}{\partial x^\alpha} dx^\alpha, \tag{1}$$

that is, they transform linearly via multiplication by the Jacobian of the coordinate transformation. Partial derivatives transform the opposite way: If f is a function, then

$$\frac{\partial f}{\partial y^\mu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial f}{\partial x^\alpha}. \tag{2}$$

That is, they transform linearly via multiplication by the inverse Jacobian. These two types of transformation are called *contravariant* and *covariant*, respectively.

Contraction

Suppose A^μ and B_μ are contravariant and a covariant vectors, respectively. Then the contraction $A^\mu B_\mu$ is unchanged by coordinate transformation, because the inverse Jacobian combines with the Jacobian to give the identity:

$$\begin{aligned} A'^\mu B'_\mu &= (\frac{\partial y^\mu}{\partial x^\alpha}A^\alpha)(\frac{\partial x^\beta}{\partial y^\mu}B_\beta) && (3)\\ &= (\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial y^\mu}{\partial x^\alpha})A^\alpha B_\beta && (4)\\ &= \delta^\beta{}_\alpha A^\alpha B_\beta && (5)\\ &= A^\beta B_\beta && (6)\end{aligned}$$

No metric is involved here.

Metric

The spacetime line element is a quadratic form in the coordinate differentials,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu. \tag{7}$$

The interval ds^2 is a scalar: it corresponds to the proper time or proper distance, a coordinate invariant concept. The metric $\eta_{\mu\nu}$ must therefore transform as a covariant tensor on each of its two indices, because only then will the Jacobians from transforming the coordinates be cancelled by inverse Jacobians. Note that $dx^\mu dx^\nu$ is symmetric under interchange of the two indices, so any antisymmetric part of $\eta_{\mu\nu}$ would drop out of ds^2, so $\eta_{\mu\nu}$ is assumed to be symmetric. In Minkowski spacetime, we can choose coordinates such that the components of $\eta_{\mu\nu}$ are $\mathrm{diag}(1,-1,-1,-1)$. These are called Minkowski coordinates, or Cartesian coordinates. The transformations that relate different Minkowski coordinate systems are Poincaré transformations: translations and Lorentz transformations. That is, if $\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\eta_{\mu\nu} = \eta_{\alpha\beta}$, it can be shown that $y^\mu = \Lambda^\mu{}_\alpha x^\alpha + b^\mu$, where $\Lambda^\mu{}_\alpha$ and b^μ are constants, and $\Lambda^\mu{}_\alpha$ is a Lorentz transformation matrix. For a proof see pp. 6-7 of my GR notes, `https://www.physics.umd.edu/grt/taj/675e/675eNotes.pdf`.

Kronecker delta

The Kronecker delta, δ^μ_ν, has one covariant and one contravariant index. It acts as the identity when either of these is contracted with a contravariant or covariant index, respectively. It is invariant under coordinate transformations.

Inverse metric

There is a unique symmetric, contravariant tensor $(\eta^{-1})^{\mu\nu}$ such that $(\eta^{-1})^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu$. We drop the $^{-1}$, and write the inverse metric simply as $\eta^{\mu\nu}$.

Index raising and lowering

Given the metric, we can lower a contravariant index by contracting with the metric, turning it into a covariant index. We can also do the opposite, contracting the inverse metric with a covariant index to make a contravariant one. We usually don't write the metric contraction explicitly, but rather simply raise or lower the index, it being understood that contraction with the metric or its inverse was used to do that.

Suppose A_μ and C_μ are covariant vectors. Then $A_\mu C^\mu = A_\mu \eta^{\mu\nu} C_\nu = A^\mu C_\mu$. That is, it doesn't matter which index is raised, because the metric is symmetric. In a Minkowski coordinate system, $C^0 = C_0$, and $C^i = -C_i$, where $i = 1, 2, 3$ are the spatial coordinates.