

4.3 Linear sigma model. The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x)$, $i = 1, \dots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

$$V(\Phi^2) = \frac{1}{2}m^2(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2$$

is a function symmetric under rotations of $\mathbf{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H ; hence, V is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, u and d . These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d) :

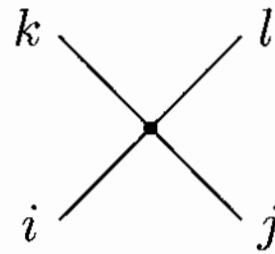
$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2) \begin{pmatrix} u \\ d \end{pmatrix}.$$

This transformation is called an *isospin* rotation. If, in addition, the strong interactions are described by a vector “gluon” field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

- (a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ . Show that the propagator is

$$\overline{\Phi^i(x) \Phi^j(y)} = \delta^{ij} D_F(x - y),$$

where D_F is the standard Klein-Gordon propagator for mass m , and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

(That is, the vertex between two Φ^1 s and two Φ^2 s has the value $(-2i\lambda)$; that between four Φ^1 s has the value $(-6i\lambda)$.) Compute, to leading order in λ , the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \rightarrow \Phi^1\Phi^2, \quad \Phi^1\Phi^1 \rightarrow \Phi^2\Phi^2, \quad \text{and} \quad \Phi^1\Phi^1 \rightarrow \Phi^1\Phi^1$$

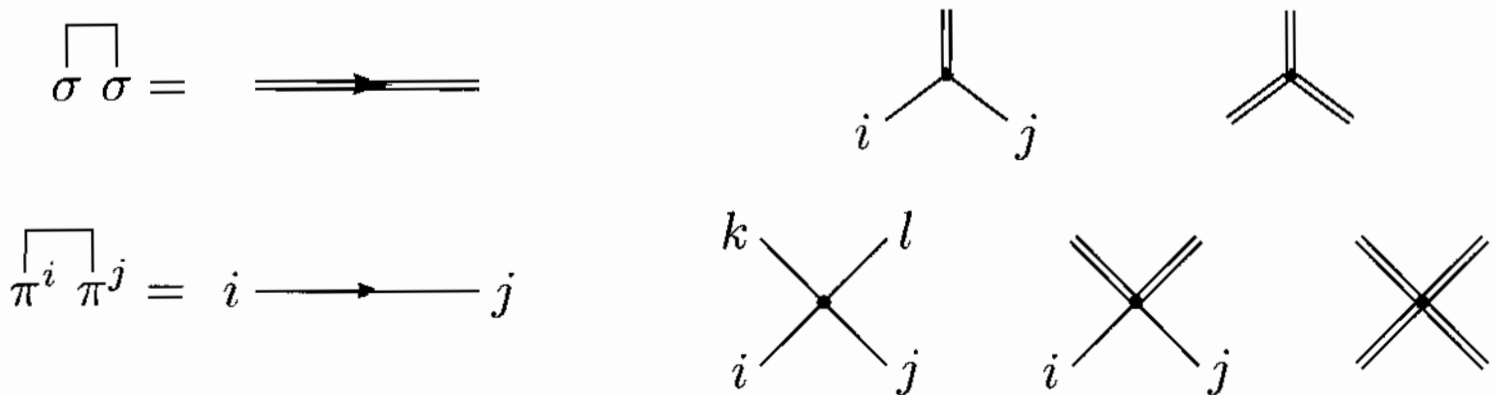
as functions of the center-of-mass energy.

Answer: The first two are $\lambda^2/(16\pi^2s)$, and the third is 9 times this. (You can adapt Schwartz (5.33) to get this.)

- (b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V . By rotational invariance, we can consider this shift to be in the N th direction. Write, then,

$$\begin{aligned} \Phi^i(x) &= \pi^i(x), \quad i = 1, \dots, N - 1, \\ \Phi^N(x) &= v + \sigma(x), \end{aligned}$$

where v is a constant chosen to minimize V . (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N - 1$ massless pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \rightarrow 0$. Construct the Feynman rules by assigning values to the propagators and vertices:

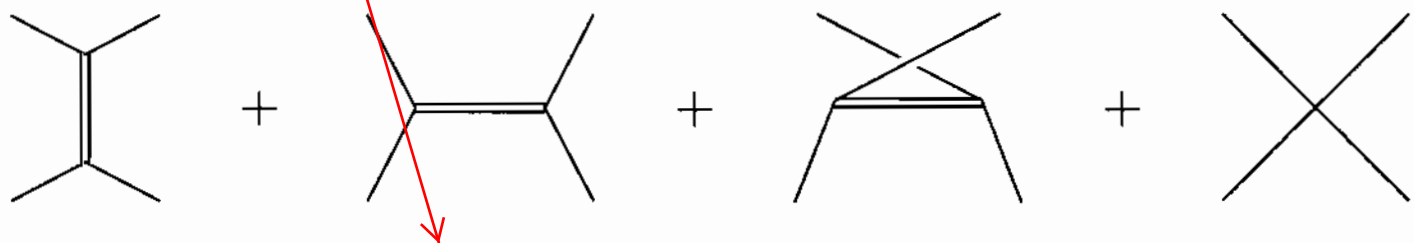


- (c) Compute the scattering amplitude for the process

$$\pi^i(p_1) \pi^j(p_2) \rightarrow \pi^k(p_3) \pi^l(p_4)$$

This "at threshold" means (because the pions are massless) that one of the 4-momenta is zero. The reason for the vanishing of the amplitude when a pion momentum vanishes is that the pions defined here, to first order, represent the symmetry variations of the field Φ , i.e. the variations that don't change the potential, so there is an underlying "shift symmetry", but it is obscured by the way Φ is expanded. See section 28.2.1 of Schwartz to see, for the case $N = 2$, how one could define the fields to make the shift symmetry manifest. Even though the symmetry is not manifest here, it remains true that a pion does not interact with anything else unless its derivative - i.e. its momentum - is nonzero.

to leading order in λ . There are now four Feynman diagrams that contribute:



Hint: Write the amplitudes in terms of Mandelstam parameters, and remember that $s+s+u=0$ for massless particles.

Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. (Hint: It may be easiest to first consider the specific process $\pi^1 \pi^1 \rightarrow \pi^2 \pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

(d) Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same nonvanishing mass.) Find the new value of v that minimizes V , and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to a .