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The Unruh Effect

The Note `https://www.physics.umd.edu/grt/taj/624c/vacuum.pdf` describes some remarkable properties of the Minkowski vacuum state of a quantum field, including the Unruh effect. This refers to the fact that the vacuum appears as a thermal state when viewed by a uniformly accelerated probe. The most general demonstration of this makes use of either axiomatic quantum field theory, or formal path integral methods. But a pretty convincing indication of it is provided by a demonstration that the 2-point correlation function of a scalar field in the vacuum satisfies the *KMS (Kubo-Martin-Schwinger) condition* as a function of proper time along any uniformly accelerated worldline, with temperature $T_U = \hbar a/2\pi$, where a is the proper acceleration.

KMS states

A thermal state at temperature β^{-1} is described by a density matrix $\rho_\beta = Z^{-1}e^{-\beta H}$, where H is the Hamiltonian of the system (and $\hbar = 1$). Let $A_t = e^{iHt}Ae^{-iHt}$ be the time-translation of A by the time t. If we let $t = -i\theta$ be imaginary, this becomes $A_{-i\theta} = e^{\theta H}Ae^{-\theta H}$. The imaginary time translation thus involves the thermal state density matrix and, for this reason, the thermal state expectation value of a product of field operators has a special property when one of the operators is translated in time by the amount $-i\beta$, viz:

$$\mathrm{Tr}[\rho_\beta A_{-i\beta}B] = Z^{-1}\mathrm{Tr}[e^{-\beta H}(e^{\beta H}Ae^{-\beta H})B] = \mathrm{Tr}[\rho_\beta BA]. \tag{1}$$

For systems with an energy spectrum bounded below by zero, but unbounded above, there is a further non-trivial property: the function $\mathrm{Tr}[\rho_\beta A_{-i\theta}B]$ is finite for $0 < \theta < \beta$, since $e^{\theta H}$ appears only in the combination $e^{(-\beta+\theta)H}$, so that contributions to the trace from arbitrarily large energy states are always suppressed. These properties, holding for all pairs of operators A and B, characterize a thermal state. Formally, the KMS condition on a "state" $\langle\rangle$ is the requirement that, for all A and B, (*i*) $\langle A_t B\rangle$ is analytic in the strip $-\beta < \mathrm{Im}\, t < 0$ of the complex t plane, and (*ii*) $\langle A_{-i\beta}B\rangle = \langle BA\rangle$.

Life in the vacuum on a hyperbola

The hyperbola $x(\eta) = (l\sinh\eta, l\cosh\eta, 0, 0)$ in Minkowski space has constant proper acceleration $a = l^{-1}$, and proper time parameter $l\eta$. (See the Note mentioned above for more details.) Consider the vacuum expectation value for the product of two scalar field operators anywhere on the hyperbola,

$$G(\eta, \eta') := \langle\Omega|\phi(x(\eta))\phi(x(\eta'))|\Omega\rangle. \tag{2}$$

Show that G satisfies the KMS condition with respect to proper time along the hyperbola, with $\beta = 2\pi l$, i.e. for a temperature $T_U = \hbar a/2\pi$.

Guidance

Insert the identity between the two field operators, using a complete set of energy-momentum eigenstates, and use the energy-momentum operator to translate the arguments $x(\eta)$ and $x(\eta')$ to the origin. (These steps are similar to those leading to the Källén-Lehmann spectral representation of the vacuum expectation value of the time-ordered product of local field operators.) Next, note that since the vacuum is boost-invariant, $G(\eta, \eta')$ depends only on $\eta - \eta'$, so you can without loss of generality set $\eta' = 0$. Next, argue that $G(\eta, 0)$ is a Lorentz-invariant (and translation invariant) function of $x(\eta)$ and $x(0)$. Next, show that in the frame in which $x(\eta) - x(0)$ has no spatial components, its time component is $2l \sinh \eta/2$. Finally, using the previous, show that $G(\eta, 0)$ is analytic in the complex η plane in the strip $-2\pi < \mathrm{Im}\, \eta < 0$, and that $G(\eta - i2\pi, 0) = G(0, \eta)$.