HW#3 —Phys624—Fall 2017 Prof. Ted Jacobson
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1. Consider a quantum particle in one dimension, with Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}m_0^2 x^2 - \tfrac{1}{4!}gx^4. \tag{1}$$

The point of this problem is to explore the analogy with aspects of quantum field theory. (I've chosen funny units to make the example look more like the field theory. So, with $\hbar = 1$ as in field theory, the free oscillator has Schrodinger position operator $x = (a + a^\dagger)/\sqrt{2m_0}$.) Let $|0\rangle$ be the ground state of the "free" theory, i.e. with $g = 0$, and let $|\Omega\rangle$ be the ground state of the "interacting" theory, with $g \neq 0$. The first excited state in the free theory has energy m_0 relative to the ground state, whereas in the interacting theory the first excited state has, in general, some other energy m relative to the interacting ground state.

(a) Evaluate the "Feynman propagator" $\langle 0|Tx(t)x(t')|0\rangle$ and its Fourier transform in the free theory, where $x(t)$ is the Heisenberg position operator.

(b) Derive the analog of the spectral representation of $\langle\Omega|Tx(t)x(t')|\Omega\rangle$, now for the interacting theory, following the steps in Weigand's notes leading to (2.27).

(c) Translate problem 2 in this homework into the equivalent problem for this 1d quantum particle, and solve all parts. To translate part (c), replace the preamble with "The spectral function can be written in the following form, where $\tilde{\rho}(M^2)$ consists of the contributions from all the higher energy states:" and define $Z := 2(E_1 - E_0)|\langle 1|x(0)|\Omega\rangle|^2$. (The factor $2(E_1 - E_0)$ is explicit, whereas in the field theory this factor is implicit in the continuum normalization of the state $|1_{\vec{0}}\rangle$.)

(d) Now change the interaction term from $\frac{1}{4!}gx^4$ to $\frac{1}{2}bx^2$, which is simple enough that you can solve the interacting theory exactly. Show explicitly that for this theory $Z = 1$, and explain how this is consistent with the general result that $Z = 1$ if and only if the theory is free.

2. **Proof that $Z < 1$ unless the theory is free** - see attached.

2. Proof that Z < 1 unless the theory is free

Problem from Timo Weigand

From the lecture we know that

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\,\rho(M^2)\,D_F(x-y,M^2)\,. \tag{12}$$

a.) Show with the definitions

$$D(x-y) := \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\,p(x-y)} \text{ and } D_F(x-y) := \theta(x_0-y_0)D(x-y)+\theta(y_0-x_0)D(y-x)$$

that

$$\lim_{x_0-y_0\to 0_+} \partial_{x_0} D_F(x-y) = -\frac{i}{2}\delta^3(\mathbf{x}-\mathbf{y}) = -\lim_{y_0-x_0\to 0_+} \partial_{x_0} D_F(x-y).$$

b.) With the canonical commutation relations

$$[\phi(t,\mathbf{x}),\dot\phi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \tag{13}$$

equation (12) and point a.), show that

$$\int_0^\infty \frac{dM^2}{2\pi}\,\rho(M^2) = 1. \tag{14}$$

Hint: Take the x^0 derivative of the lhs of (12), and subtract its limit as $x^0 - y^0$ approaches zero from above from its limit as $x^0 - y^0$ approaches zero from below. This will produce the commutator in (13).

c.) For a theory with just one kind of one-particle state and no bound states, the spectral function takes the following form:

$$\rho(M^2) = (2\pi\,Z)\delta(M^2-m^2) + \tilde\rho(M^2)\,. \tag{15}$$

Show with (15) and (14) that $Z \le 1$ and that $Z = 1$ if and only if $\tilde\rho(M^2) = 0$.

d.) Show that $\tilde\rho(M^2) = 0$ implies $(\partial^2 + m^2)\phi(x)|\Omega\rangle = 0$. Now, the Reeh-Schlieder theorem for a *local* operator $\mathcal{O}$ states that $\mathcal{O}|\Omega\rangle = 0$ implies $\mathcal{O} = 0$.[2] Thus, $Z = 1$ if and only if the field $\phi(x)$ is a free field.

Hint: Show that the inner product of this expression with every state in a basis of energy-momentum eigenstates vanishes.

[2] This is not in contradiction with $P_\mu|\Omega\rangle = 0$ because P_μ involves integrals.