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There is a 1-1 correspondence of vectors $x^\mu = (t, \vec{x})$ in Minkowski space with 2×2 hermitian matrices, $X = tI + \vec{x}\cdot\vec{\sigma}$, and $\det X = t^2 - \vec{x}\cdot\vec{x}$ is the squared norm of the Minkowski vector. For any 2×2 complex matrix Λ, $X \to \Lambda^\dagger X \Lambda$ is a linear transformation taking X to another hermitian matrix with determinant $|\det\Lambda|^2 \det X$. If $\det\Lambda = 1$, this is a Lorentz transformation of x^μ. Moreover, the transformation corresponding to the matrix product is the composition of two such Lorentz transformations in the reverse order: $(\Lambda_2\Lambda_1)^\dagger X(\Lambda_2\Lambda_1) = \Lambda_1^\dagger(\Lambda_2^\dagger X\Lambda_2)\Lambda_1$, so this correspondence yields an anti-homomorphism $SL(2,C) \to SO^+(3,1)$. The "+" indicates the component connected to the identity, which is the component that preserves the direction of time and the orientation of space. This is a $2\to 1$ cover, because Λ and $-\Lambda$ map to the same Lorentz transformation.[1]

There are two distinct representations of $SL(2,C)$, sometimes called "right" and "left" handed:

$$\Lambda_R = \exp[\tfrac{1}{2}(\vec{\eta} + i\vec{\theta})\cdot\vec{\sigma}], \qquad \Lambda_L = \exp[\tfrac{1}{2}(-\vec{\eta} + i\vec{\theta})\cdot\vec{\sigma}], \tag{1}$$

which satisfy

$$\Lambda_R^\dagger \sigma^\mu \Lambda_R = \Lambda^\mu{}_\nu \sigma^\nu, \qquad \Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu \tag{2}$$

where $\sigma^\mu = (I, \vec{\sigma})$, $\bar{\sigma}^\mu = (I, -\vec{\sigma})$, and $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is a Lorentz transformation with boost angle $\vec{\eta}$ and rotation angle $\vec{\theta}$. Note that the left representation differs from the right representation only by the sign of the boost angle term.

1. *Lorentz invariance of Weyl, Dirac, and Majorana Lagrangians*

   (a) Verify (2) for the two special cases (i) pure rotation through θ about the z axis, and (ii) pure boost through η in the z direction.

   (b) Show that the Weyl Lagrangians $iR^\dagger\sigma^\mu\partial_\mu R$ and $iL^\dagger\bar{\sigma}^\mu\partial_\mu L$ are Lorentz invariant, if R and L are right and left handed spinors, respectively. That is, show that $R^\dagger\sigma^\mu\partial_\mu R = R'^\dagger\sigma^\mu\partial'_\mu R'$, where $\partial'_\mu = \partial/\partial x'^\mu$ and $R' = \Lambda_R R$, and similarly for the left-handed spinor.

   (c) Show that the Dirac mass term $-m(R^\dagger L + L^\dagger R)$ is Lorentz invariant.

   (d) Show that $\sigma_2\vec{\sigma}\sigma_2 = -\vec{\sigma}^*$, where σ_2 is the Pauli matrix otherwise known as σ^y.

   (e) Show that $i\sigma_2 R^*$ transforms as a left-handed spinor, if R is a right-handed one.

   (f) The previous three parts imply Lorentz invariance of the Majorana mass term, $-i(M/2)(R^T\sigma_2 R - R^\dagger\sigma_2 R^*)$. (This term would vanish if the components of R were commuting fields, but if the components are fermionic, and hence anticommute with each other, it is nonvanishing.) Show that it is Lorentz invariant just using $\det\Lambda = 1$ directly, without any appeal to the previous parts.

[1] The group $SL(2,C)$ is the simply connected covering group of $SO^+(3,1) = SL(2,C)/Z_2$. All closed loops are contractible in $SL(2,C)$, whereas $SO^+(3,1)$ has noncontractible closed loops. To be specific, there is one homotopy class of noncontractible loops based at any point. For example, the rotations through angles $\theta \in [0, 2\pi]$ form a noncontractible closed loop in the group, whereas the rotations through angles $\theta \in [0, 4\pi]$ form a contractible loop.

2. *Seesaw mechanism*

Consider the Lagrangian for two, 2-component spinor fields coupled by a Dirac mass term, and with a Majorana mass term for one of the fields:

$$\mathcal{L} = iR^\dagger \sigma^\mu \partial_\mu R + iL^\dagger \bar{\sigma}^\mu \partial_\mu L - m(R^\dagger L + L^\dagger R) - i(M/2)(R^T \sigma_2 R - R^\dagger \sigma_2 R^*) \qquad (3)$$

(a) Re-express the Lagrangian in terms of the doublet of left-handed fields, $N = \begin{pmatrix} L \\ \chi \end{pmatrix}$, where $\chi = i\sigma_2 R^*$, with a term involving a mass matrix of the form $\hat{M} \otimes i\sigma_2$. That is $\hat{M}$ is a 2×2 matrix whose entries are multiplied by $i\sigma_2$, yielding a 4×4 matrix. You will need to drop a total divergence. (*Hint*: After integrating by parts on the R kinetic term, rewrite it in terms of its transpose, and remember that the fermion fields anticommute.)

(b) Write the field equation for N. (*Hint*: When finding the field equation for a complex field, you can pretend that you can vary the field and its complex conjugate separately, since the two equations that yields are together equivalent to the two equations that arise from varying separately the real and imaginary parts of the field.)

(c) Show that if N satisfies its first order field equation, and is an eigenvector of the mass matrix $\hat{M} \otimes I$, then N satisfies the Klein-Gordon equation with mass equal to the corresponding eigenvalue. (*Hint*: Use the nice identity $\sigma^{(\mu}\bar{\sigma}^{\nu)} = \eta^{\mu\nu} I$.)

(d) Find the eigenvalues of the mass matrix $\hat{M}$. Then assume $m \ll M$, and express the eigenvalues to leading order in m/M. You should find at leading order that one eigenvalue is M and the other is m^2/M. This is the *seesaw* mechanism: as M goes up, the light neutrino mass goes down.

(e) If $m = 100$GeV, what value of M would yield a light neutrino mass of 0.1eV, and what would be the heavy neutrino mass?

(f) Schwartz 11.9d, 5th printing, or e, earlier printings (second to last part of 11.9).

(g) Schwartz 11.9e 5th printing, or f, earlier printings (final part of 11.9).