HW#1 —Phys624—Fall 2017 Prof. Ted Jacobson
due at beginning of class, Thursday 09/08/16 Room PSC 3151, (301)405-6020
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1. Consider the single particle state $|\psi\rangle = \int \frac{d^3p}{(2\pi)^3}\, \psi(\vec{p},t)\, a_p^\dagger|0\rangle$ defined by a momentum space wavepacket $\psi(\vec{p},t)$.

   (a) Show that $\langle\psi|\vec{P}|\psi\rangle = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\,|\psi(\vec{p},t)|^2$, where $\vec{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a_p^\dagger a_p$ is the field momentum operator. This shows that $\psi(\vec{p},t)$ corresponds to $(2\pi)^{3/2}$ times the normalized momentum space wavefunction in quantum mechanics. (For convenience, first compute $\left[\vec{P}, a_{\vec{p}}^\dagger\right]$.)

   (b) Impose the field theory Schrodinger equation $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$ using the Hamiltonian (W1.76) (the W refers to Weigand's lecture notes), and deduce the one-particle Schrodinger equation satisfied by $\psi(\vec{p},t)$. Show that when $|\vec{p}| \ll m$ this becomes the non-relativistic, momentum space Schrodinger equation for a particle in a constant potential.

2. Consider the vacuum "equal time 2-point correlation function" for a massive scalar field,

$$\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle. \tag{1}$$

   On grounds of translational symmetry, this must depend on $\vec{x}$ and $\vec{y}$ only via the distance between them, $|\vec{x}-\vec{y}|$.

   (a) Express (1) as a one-dimensional integral over a function of momentum, m, and $|\vec{x}-\vec{y}|$. Show that it diverges when $\vec{x}=\vec{y}$. This means that there are infinitely large fluctuations in the field, which are strongly correlated at nearby points. Show that it isn't well-defined when $\vec{x}\neq\vec{y}$, because the integral doesn't converge. In the rest of this problem, we'll see that this lack of convergence isn't really important physically, and that there really is a well defined correlation function.

   (b) Insert a convergence factor $e^{-\epsilon p}$ in the integral, where ϵ is a positive real number. Show that now the integral is well defined, and that in the limit $\epsilon \to 0$ the result for the massless case is is

$$\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle = (2\pi)^{-2}|\vec{x}-\vec{y}|^{-2}. \tag{2}$$

   Explain how, apart from the numerical factor, this form follows from translational symmetry and dimensional analysis.

   (c) Find the $\epsilon \to 0$ limit for the case $m \neq 0$. (The result involves a modified Bessel function: $mK_1(md)/4\pi^2 d$, where $d := |\vec{x}-\vec{y}|$.) Show that for small $|\vec{x}-\vec{y}|$ your result is the same as the massless case, and for large $|\vec{x}-\vec{y}|$ it decays exponentially, with a decay rate determined by the mass. This means that at distances much smaller than the Compton wavelength, the mass makes no difference, while at distances much larger than the Compton wavelength, the correlation is exponentially small.[1]

[1] A great resource for all kinds of information about special functions is the NIST Digital Library of

(d) **This part is optional; but in any case read it, to learn about the results.** "Smear," i.e. average the field operators in (1) by integrating them against normalized Gaussian weighting functions $(2\pi\sigma^2)^{-3/2}e^{-|\vec{x}-\vec{x_0}|^2/2\sigma^2}$, and similarly for y.

i. Show that the smeared correlation function is well-defined for any $\sigma \neq 0$.

ii. Set $m = 0$ and show that as $\sigma/|\vec{x_0} - \vec{y_0}| \to 0$ you recover the previous result, $(2\pi)^{-2}|\vec{x_0} - \vec{y_0}|^{-2}$ without neglecting anything.

iii. Set $m = 0$ and $\vec{x_0} = \vec{y_0}$, and show that the correlation function (which is then just the mean square smeared field operator) becomes $(8\pi^2)^{-1}\sigma^{-2}$. That is, the size of the fluctuations is completely controlled by the size of the smearing region.

Mathematical Functions, `http://dlmf.nist.gov/`. Our integral is close to the integral representation of K_1 in Section 10.32, integral 10.32.7. But taking the limit using a change of variables and this integral representation is still tricky. I don't know a valid, simpler way, but perhaps there is one. Peskin and Schroeder obtain a convergent integral by expanding the integration to $p = -\infty$, and then deforming the contour to wrap around the branch cut along the positive imaginary axis. The latter integral is indeed well defined, however I don't believe the deformation is justified by the behavior at infinity, so this really amounts to a definition of the integral. There may be a physical and/or mathematical justification of their definition, but I don't know what it is.