Why a “filled shell” has zero angular momentum

The antisymmetry of the state implies that the total angular momentum vanishes. Let’s start with the simplest example, two spin-$\frac{1}{2}$ states. We can form a two-particle state with $S_z = 0$ as $|\frac{1}{2}\rangle|-\frac{1}{2}\rangle$, or as $|-\frac{1}{2}\rangle|\frac{1}{2}\rangle$, but only the antisymmetric linear combination $(|\frac{1}{2}\rangle|-\frac{1}{2}\rangle - |-\frac{1}{2}\rangle|\frac{1}{2}\rangle)/\sqrt{2}$. has vanishing total angular momentum.

To see why this works in general, consider the totally antisymmetric combination of products of the $2j+1$ states of a spin-j representation (i.e. the Slater determinant of these states). Now act with any rotation on this state. Each of the states in the product will be acted on by the same rotation operator, and the resulting state will still be totally antisymmetric. But there is only one totally antisymmetric tensor product of n vectors in an n-dimensional vector space, so after the rotation the state is the same, up to a possible scalar multiple. In fact the scalar multiple is nothing but the determinant of the 1-particle rotation matrix, which is 1. This means the antisymmetric state is invariant under all rotations. The angular momentum components are the infinitesimal generators of rotations, so the fact that the state is rotationally invariant implies that $\vec{J}$ annihilates it.

Another way to see that in addition to J_z, both J_x and J_y annihilate the totally antisymmetric state, is to act with $J_\pm = J_x \pm iJ_y$ on the state. Every term either vanishes because $m = j$ can’t be raised, or because $m = -j$ can’t be lowered, or because the result of the action is to make two m values the same, which yields zero because of the antisymmetry of the state.