HW#2—Phys623—Spring 2018 Prof. Ted Jacobson
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1. Analyze the effect of an external electric field on the $2s_{1/2}$ and $2p_{1/2}$ levels of hydrogen, taking into account that these levels are split by the Lamb shift by an amount 1.06 GHz. The other levels, and the hyperfine structure, can be ignored for the purpose of this problem, so this is only a two-level system. Show that the energy shifts are quadratic in the field strength for small amplitude electric fields, but that they become linear at larger strengths. Thus there is a threshold field strength, call it $\mathcal{E}_0$, where the behavior changes from quadratic to linear. Estimate $\mathcal{E}_0$ in Volts/cm. (I found $\simeq 240$ V/cm.) Find and sketch the energy levels as a function of the electric field strength $\mathcal{E}$. Explain how this problem provides an example of Kramers degeneracy (and use this fact to cut your work in half). In computing the required matrix elements, you may construct the $2s_{1/2}$ and $2p_{1/2}$ states using the nonrelativistic, hydrogen wave functions, for which $\langle 2p, m_\ell = 0|z|2s\rangle = -3a_0$. You'll probably need a CG coefficient. If so, derive this coefficient from scratch, i.e. by finding the relevant combination of product states using the lowering operator method (or any other method you devise—and please let me know if you devise another method). Check your CG coefficient against a table/book/Mathematica/etc.

2. Write the $2p_{1/2}(F = 0)$ state of hydrogen in terms of the product kets $|2p, m_l\rangle|m_s\rangle|m_I\rangle$. (The 1/2 subscript denotes the eigenvalue of $\vec{J} = \vec{L} + \vec{S}$, and $\vec{F} = \vec{J} + \vec{I}$ is the total angular momentum of the atom, where $\vec{I}$ is the nuclear spin.) Use two methods, and compare: i) first find the $J = 1/2$ states of the electron, and then combine with the nuclear spin states to find the $F = 0$ state; ii) list all the product kets in the L, S, I basis that have $m_F = 0$, and then determine the coefficients of the unique (normalized) superposition of these that is invariant under all rotations. (*Hint*: Use F_+.)

3. In classical physics, the magnetic moment of a spinning axisymmetric body with a uniform ratio of charge density to mass density is given by $\mu = (q/2mc)L$, where q is the charge, m is the mass, and L is the orbital angular momentum. The "g-factor" for a quantum particle with total angular momentum J is defined by the equation $\mu = g(q/2mc)J$. For intrinsic spin of an electron or proton, $J = S = 1/2$. For a compound nucleus the spin is labeled by the letter I, and can take different values.

The Dirac equation implies that the g-factor for the electron is $g_e = 2$. This result can also be obtained from the nonrelativistic limit of the Dirac Hamiltonian. For a charge e coupled to an electromagnetic vector potential $\vec{A}(x)$ this nonrelativistic Hamiltonian for the two-component wave function is $H = [\vec{\sigma} \cdot (\vec{p} - \frac{e}{c}\vec{A})]^2/2m$. Show that this implies $g_e = 2$. (*Hint*: It is convenient to use index notation, and $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$.)

(*continued...*)

4. Qualifier Problem II-4 August 2013: Symmetries and Selection Rules.

Let H_0 be the Hamiltonian of a spinless nonrelativistic particle in a central potential $V_0(r)$. Energy eigenstates are the eigenstates of angular momentum and can be labeled as $|n, l, m\rangle$. Apart from the $(2l+1)$-fold degeneracy arising from spherical symmetry, there are no other degeneracies in the spectrum of H_0. This problem concerns how the spectrum changes when a perturbation V of the following form is added to H_0:

$$V(r, \theta, \phi) = \lambda\, e^{-r^2/R^2}\, r^K\, Y_K^0(\theta, \phi).$$

Here the perturbing potential is written in spherical coordinates, the parameter λ represents the strength of the potential, R is a constant with units of length, K is a positive integer, and Y_l^m is a spherical harmonic.

When doing the problem, you may find helpful to think about parity, time reversal, and spherical tensors. Assume that quantities never vanish accidentally, i.e. they only vanish for reasons of symmetry. If Clebsch-Gordan coefficients arise, simply state your result in terms of them and do not evaluate them explicitly.

In Parts (a) and (b), work to all orders in the strength λ of the perturbation.

(a) When the perturbation is included, does the quantum number m remain a good quantum number for the system? Explain briefly why or why not.

(b) The perturbation breaks the $(2l+1)$-fold degeneracy. Is there any surviving degeneracy in the spectrum? If not, explain briefly why not. If so, identify the degeneracy and explain briefly its origin.

In Parts (c)-(e), assume that the strength parameter λ is small. Let us denote by $\Delta E_{n,l,m}$ the energy shift of the state $|n, l, m\rangle$ (as labeled in the unperturbed eigenbasis) due to the perturbation. For small λ, one might generically expect that the energy shift is given by the first-order perturbation theory, so that $\Delta E_{n,l,m} = \langle n, l, m|V|n, l, m\rangle \propto \lambda$. However, under certain circumstances, the first-order perturbation vanishes, and the dominant behavior is quadratic in λ, i.e. $\Delta E_{n,l,m} \propto \lambda^2$.

(c) Consider the case $K = 3$, where $V = \lambda\, e^{-r^2/R^2}\, r^3\, Y_3^0(\theta, \phi)$. Determine which energy levels, if any, have a quadratic dependence on λ as the leading behavior. (*Hint:* This amounts to asking for which states the first-order perturbation vanish. Consideration of parity may be helpful.)

(d) Consider the case $K = 6$, where $V = \lambda\, e^{-r^2/R^2}\, r^6\, Y_6^0(\theta, \phi)$. Using the Wigner-Eckart theorem, determine which energy levels, if any, have a quadratic dependence on λ as the leading behavior.

(e) Consider the case $K = 2$, where $V = \lambda\, e^{-r^2/R^2}\, r^2\, Y_2^0(\theta, \phi)$. In this case, $\Delta E_{n,l=1,m} \propto \lambda$ for the states with $l = 1$. You may assume this linear dependence to be correct without proving it. Using the Wigner-Eckart theorem, express the ratio of the energy shifts $\Delta E_{n,l=1,m}/\Delta E_{n,l=1,m'}$ in terms of Clebsch-Gordan coefficients.

(f) (*Additional part not on the qualifier*) Evaluate the CG coefficients and ratios from part (e) using whatever method you like, e.g. computed from scratch by hand, using a table of CG coefficients, or using Mathematica.