HW#9 —Phys622—Spring 1998 Prof. Ted Jacobson
Due Friday, April 17, 1998 Room 4117, (301)405-6020
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1. The wavefunction for all the electrons in the universe must be antisymmetric under interchange of any two electrons...for example one on earth and one on Mars, or one in your body and one in a rock on the top of Mt. Everest. Explain why it is safe to ignore this fact. Formulate a precise statement of when (i.e., for which observations) the symmetry or antisymmetry of identical particle wavefunctions can be neglected.

2. A white dwarf star is held up by electron degeneracy pressure. Treating the electrons as free particles in a box, the pressure is given by $p = \alpha n^{\gamma}$, where α and γ are constants and n is the electron density. Show that γ and α are given by (i) $\gamma = 5/3$ and $\alpha = \hbar^2(3\pi^2)^{2/3}/5m$ using the nonrelativistic energy $E = p^2/2m$, and by (ii) $\gamma = 4/3$ and $\alpha = \hbar c(3\pi^2)^{1/3}/4$ using the relativistic energy $E = cp$.

[When the density gets sufficiently high ($n \gg (mc/\hbar)^3$) the Fermi energy becomes relativistic, and one obtains a lower pressure for a given density. This was a key ingredient in Chandrasekhar's 1930 derivation of a maximum mass for white dwarfs, the so-called *Chandrasekhar limit.* A similar limit also applies to neutron stars, which have nuclear densities and are held up by neutron degeneracy pressure.]

3. Problem 17.1, Schwabl. (Finding the spectrum of the isotropic oscillator using cartesian and polar coordinates.) In part (b) use the series solution method, which is explained in detail for the Coulomb potential in section 6.3. Add a final part, (c), in which you compare (a) and (b) and show that they yield the same energy levels and degeneracies.

4. If $\psi = Ae^{-r/a_0}$ is an eigenstate with energy E for a particle of mass m in a potential $V(r)$ that vanishes as $r \to \infty$, what are E and $V(r)$?

5. How high must the temperature of Hydrogen gas be if a substantial fraction of the atoms are in an excited state?

6. Compute the probability current density for the $2s$ and $2p$ states of Hydrogen for all allowed values of m. Interpret your results in terms of the classical electron orbits if possible. (That is, which orbits should be "superposed" to produce these currents?)

7. *Adapted from Physics Qualifier, fall 1987 and spring 1983, Problem II-2.*

Proton and neutron interact via an attractive nuclear force of a very small range r_0 and form an s-wave bound state (called the *deuteron*) with an energy E_0. Suppose that

$$|E_0| \ll \frac{\hbar^2}{m r_0^2}, \tag{1}$$

where m is the mass of the proton or neutron (their masses are almost equal).

Show that for $r > r_0$ the wave function of their relative motion ($\mathbf{r} = \mathbf{r}_p - \mathbf{r}_n$) can be approximately written as:

$$\psi(r) = \sqrt{\frac{k}{2\pi}}\frac{1}{r}\, e^{-kr}, \tag{2}$$

where $\hbar^2 k^2/m = |E_0|$. What does the condition (1) tell you about the wave function?

8. Consider an atom with the nuclear charge Ze and Z electrons.

(a) Neglecting interaction between electrons, and filling up the energy shells consecutively by Z electrons, evaluate roughly the dependence of the maximal principal quantum number N on Z for large Z.

(b) Let us define the radius of the atom as the location of the maximum of the radial probability density for the state with principal quantum number N and maximal angular momentum $l = N - 1$. Using the result on page 131 in section 6 of Schwabl, evaluate how the radius of the atom depends on Z. Does the answer make physical sense? Explain.

[In fact, the size of a real atom, with electron interactions taken into account, grows very slowly with Z, I think as $Z^{1/6}$. Nevertheless, most of the electrons are at a radius that scales as $Z^{-1/3}$, in agreement with this simple-minded exercise.]

9. **(a)** Suppose a Hamiltonian depends on a parameter λ. Then, the eigenvectors and eigenvalue also depend on λ:

$$\hat{H}(\lambda)|n,\lambda\rangle = E_n(\lambda)|n,\lambda\rangle. \tag{3}$$

Show that

$$\frac{\partial E_n(\lambda)}{\partial \lambda} = \left\langle n,\lambda \left| \frac{\partial \hat{H}(\lambda)}{\partial \lambda} \right| n,\lambda \right\rangle, \tag{4}$$

where the parameter n labels different eigenstates and eigenvalues of a discrete spectrum, and the eigenvectors are normalized.

Eq. (4) is valid not only for a Hamiltonian but for any Hermitian operator and its eigenvalues and eigenfunctions.

(b) The radial Schrödinger equation (Eq. (6.11) of Schwabl) with a given l has a discrete bound state spectrum. We can label its energy levels consecutively in the order of increase by an integer called the *radial quantum number* n_r: $E_{n_r l}$. For example, for $l = 5$, the lowest energy level is E_{15}, the first excited E_{25}, etc. Thus, by definition, $E_{n_r l}$ increases with increasing n_r at fixed l.

How do the energy levels $E_{n_r l}$ change with increasing l at fixed n_r? (*Hint*: Use part (a) of this problem.)

(c) A particle of zero spin is subject to a spherically symmetric potential. Answer the following questions and compare with the cases of the three dimensional isotropic oscillator and the Coulomb potential. (*Hint*: Use part (b) of this problem.)

i. May energy levels of the particle have the degeneracy $g = 2$?

ii. What are the possible values of degeneracy g of the first excited energy level?

iii. A quantum level has the degeneracy $g = 7$. What are the possible values of the angular momentum l of this level? Answer the same question also in the case where $g = 9$.

Let us label the energy levels of the particle E_N in the order of increase by the integer N starting from $N = 1$ (regardless of n_r and l).

iv. What is the maximal possible value of angular momentum l in a state $|N\rangle$?

v. What is the maximal possible degeneracy g of a state $|N\rangle$?