HW#8 —Phys622—Spring 1998 Prof. Ted Jacobson
Due Monday, April 6, 1998 Room 4117, (301)405-6020
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1. Show that for eigenstates of J_z the expectation values of J_+, J_-, J_x and J_y vanish, and the expectation values of J_x^2 and J_y^2 are equal.

2. Evaluate both sides of the angular momentum uncertainty relation

$$\Delta J_x \Delta J_y \geq |\langle [J_x, J_y] \rangle|/2 \tag{1}$$

in the state $|jm\rangle$. Show that when $m = \pm j$ the inequality becomes an equality. Show that when $m = \pm j$ the conditions for a minimum uncertainty state for the operators J_x and J_y are satisfied.

3. Show that $J_\pm |jm\rangle = \sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle$.

4. Use problem 3 to find the matrices representing J_x, J_y, J_z for $j = 0, \frac{1}{2}, 1$ in the basis $\{|jm\rangle\}$. Check by matrix multiplication that your matrices satisfy the commutation relations $[J^i, J^j] = i\hbar\epsilon^{ijk} J^k$.

5. **(a)** Show that the vector operators $\mathbf{r}$, $\mathbf{p}$, $\mathbf{r} \times \mathbf{p}$ all satisfy

$$[L^i, V^j] = i\hbar\epsilon^{ijk}\, V^k \tag{2}$$

(which is equivalent to $[\mathbf{n} \cdot \mathbf{L}, \mathbf{V}] = i\hbar \mathbf{V} \times \mathbf{n}$ for all vectors $\mathbf{n}$).

(b) Show that the dot product of any two operators satisying (2) *commutes* with $\mathbf{L}$.

(c) Show that all scalar functions $S(\mathbf{r}, \mathbf{p})$ of $\mathbf{r}$ and $\mathbf{p}$ commute with $\mathbf{L}$, and all vector functions $\mathbf{V}(\mathbf{r}, \mathbf{p})$ satisfy (2).

6. Find the spherical harmonic $Y_{ll}(\theta, \phi)$ by solving the differential equation $L_+ Y_{ll} = 0$, with L_+ given by eqn. (5.18d) of Schwabl. Find the normalization factor so that $\int d\Omega Y_{ll}^* Y_{ll} = 1$.

7. Problem 5.6, *Schwabl* $(L^2|\psi\rangle = 2\hbar^2|\psi\rangle,\ L_x|\psi\rangle = \hbar^2|\psi\rangle)$

8. Show that reflection across the x-y plane is the same as parity ($\vec{r} \to -\vec{r}$) followed by 180° rotation about the z axis.

9. At sufficiently low energy it makes sense to consider the rotation of a molecule as if it were a rigid body whose configuration is described by a pair of angles or a unit vector. The Hamiltonian for the rotational motion takes the form

$$H = \frac{J_u^2}{2I_A} + \frac{J_v^2}{2I_B} + \frac{J_w^2}{2I_z}$$

where u, v, w are the coordinates with respect to the principal axes of the moment of inertia, and the operators J_u, J_v, and J_w are the angular momentum operators in the body-fixed coordinate system, which satisfy the usual commutation relations of angular momentum operators (up to a minus sign on the right hand side that can be eliminated by changing the sign of the J's). [*Note*: I do not understand this business about the minus sign. It is discussed (obscurely) in Landau & Lifshitz, §103.]

(a) Why can the energy eigenstates be assumed to be eigenstates of $J^2 = J_u^2 + J_v^2 + J_w^2$?

(b) What is the energy spectrum and degeneracy for a "spherical top", i.e. a body with $I_A = I_B = I_C$?

(c) What is the energy spectrum and degeneracy for a "symmetric top", i.e. a body with $I_A = I_B \neq I_C$?

(d) What is the energy spectrum and degeneracy for an "asymmetric top", i.e. a body with $I_A \neq I_B \neq I_C \neq I_A$ in the sector with the lowest non-zero total angular momentum?

10. At low enough temperature the conduction electrons in a metal behave as a degenerate Fermi gas. They can't all line up opposite to the magnetic field unless half of them raise their momentum, but that would give them more energy. Thus the paramagnetic susceptibility at low temperature is determined purely by the exclusion principle, and is independent of temperature.

(a) The low-field susceptibility per unit volume χ_0 is defined by $\chi_0 = \lim_{B\to 0}(M/VB)$, where M is the total magnetic moment, V is the volume, and B is the magnetic field. Show that, for a gas of noninteracting spin-$\frac{1}{2}$ fermions, neglecting the effect of B on the orbital motion, χ_0 is given at zero temperature by

$$\chi_0 = \tfrac{3}{2}n\mu^2/E_F = \mu^2 D(E_F), \tag{3}$$

where μ is the intrinsic magnetic moment of the particles, n is the particle density, E_F is the Fermi energy, and $D(E_F)$ is the density of states at the Fermi surface.

Hint: Use your results from problem 5(b) of Homework #7. Consider the fermions with spin along and opposite to $\vec{B}$ as two separate populations, with one common Fermi energy. (Why is the Fermi energy the same for both populations?) Make the approximation $\mu B \ll E_F$.

(b) Suppose the particles are electrons.

i. Roughly how small must B be in order for eqn. (3) to hold?

ii. At finite temperature, if $kT \ll E_F$, the correction to (3) is of order $(kT/E_F)^2$. What is the order of magnitude of this thermal correction at room temperature for E_F equal to the Fermi energy of copper?