HW#7 —Phys622—Spring 1998 Prof. Ted Jacobson
Due Friday, March 20, 1998 Room 4117, (301)405-6020
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Read section 6.4 (*The Two-Body Problem*) and section 13.1 (*Identical Particles*). Also read section 3.7 (*The Potential Well, Resonances*) although we will not cover it in class.

1. Two identical particles of mass m interact with each other via the three-dimensional harmonic potential $V(\mathbf{r}_1 - \mathbf{r}_2) = k(\mathbf{r}_1 - \mathbf{r}_2)^2/2$. Find the energy spectrum and the eigenfunctions of the system, when the particles are

i. bosons,
ii. fermions.

2. Consider two particles in one dimension. One particle occupies a state described by a wave function $\psi_1(x)$; the other particle the state described by a wave function $\psi_2(x)$. The wave functions $\psi_1(x)$ and $\psi_2(x)$ are normalized, orthogonal, and have opposite parity:

$$\psi_1(-x) = \psi_1(x), \qquad \psi_2(-x) = -\psi_2(x). \tag{1}$$

($\psi_1(x)$ and $\psi_2(x)$ could be the first and the second energy eigenfunctions of an oscillator.)

Answer the following questions in the cases where the particles are

i. not identical,
ii. identical bosons,
iii. identical fermions.

(a) Write the normalized wave function of the system, $\psi(x_1, x_2)$, in terms of $\psi_1(x)$ and $\psi_2(x)$.

(b) Integrating the wave function $\psi(x_1, x_2)$ found in Problem 2a, calculate the probabilities that

i. both particles are located in the right semi-space, $x > 0$;
ii. both particles are located in the left semi-space, $x < 0$;
iii. the particles are located in different semi-spaces.

Make sure that these three probabilities add up to unity.

What is the difference in the behavior of bosons and fermions?

3. *Adapted from Physics Qualifier, January 1982, Problem II-1.*

Consider a quantum particle of mass m moving in a three-dimensional spherically-symmetrical attractive potential that vanishes everywhere except at $r = d$:

$$U(r) = -\lambda\delta(r - d), \qquad \lambda > 0. \tag{2}$$

Consider a spherically-symmetrical bound state of the particle in this potential.

(a) Find a transcendental equation for the energy of the bound state. *Suggestion*: This is probably simplest if you first write the Schrödinger equation in the position representation, replace $\psi(\mathbf{r})$ by $\psi(d)$ in the potential term, and then Fourier transform the resulting equation.

(b) From the equation for the bound state energy, find the minimal value λ_{min} of λ for which the bound state exists. Conclude that a sufficiently weak potential well in three dimensions does not have a bound state, unlike in one- and two-dimensional cases.

(c) For $\lambda \gg \lambda_{\text{min}}$, solve (approximately) the equation for the bound state energy and find the energy explicitly. Note that the width of the well, d, drops from the result in this limit. Compare the energy with the bound state energy of the one-dimensional delta-function potential.

4. Let us modify the previous Problem 3 to the extent that the wave function $\phi(\mathbf{p})$ of the particle is subject to the following boundary condition in momentum space:

$$\phi(\mathbf{p}) = 0, \qquad \text{when } |\mathbf{p}| \leq p_0, \qquad p_0 > 0. \tag{3}$$

That is, the particle cannot occupy the states with momenta smaller than p_0. In this case, a bound state is a state that has an energy lower than $E_0 = p_0^2/2m$.

(a) Show that, if condition (3) is imposed, the potential well (2) has a bound state, no matter how small λ is.

(b) Find (approximately) the energy of the bound state in the case where the attractive potential is weak:

$$\lambda \ll \hbar p_0/m. \tag{4}$$

(c) In case (4), compare the energy of the bound state with the energy level E_0.

[*In this problem, p_0 mimics the effect of the Fermi surface in a metal. Inside the Fermi surface ($p < p_0$) all quantum states are occupied by other electrons, so the wave function of a probe electron must obey condition (3). Interestingly, the introduction of the Fermi surface allows an arbitrarily weak attractive potential to produce a bound state, which is not the case in three dimensions without the Fermi surface. In reality, an effective attractive potential acts between the electrons and binds them in pairs. This phenomenon, which is an important ingredient of the theory of superconductivity, was discovered by Leon Cooper, who is C in BCS.*]

5. *Adapted from Physics Qualifier, Fall 1989, Problem II-5.*

A large number N of non-interacting, indistinguishable, spin-1/2 fermions of mass m (electrons) is constrained to a large cube of volume $V = L^3$, so that the particle density is $n = N/L^3$.

(a) i. What are the allowed wave vectors $\mathbf{k}$ for the one-particle states, and what is the density of states in the $\mathbf{k}$-space?

ii. Calculate the energy density of states $D(E) = dN/dE$ as a function of energy E.

(b) In the ground state, all the states with $k < k_F$ and are occupied, and all the states with $k > k_F$ are empty, where k_F is called the Fermi vector and $E_F = \hbar^2 k_F^2/2m$ is called the Fermi energy.

Express the following quantities in terms of the electron density n and mass m:

i. k_F,

ii. E_F,

iii. the density of states at the Fermi energy, $D(E_F)$.

iv. Evaluate E_F in eV for the density of electrons in copper, $n = 8.5 \times 10^{22}$ cm^{-3} (and $m = 9.1 \times 10^{-28}$ grams).

(c) Suppose the system is placed in a weak, slowly varying in space electric potential $\phi(\mathbf{r})$. (The fermions have the electric charge e.) Find the local deviation of the electron density $\delta n(\mathbf{r})$ from the equilibrium density n in response to the given electric potential $\phi(\mathbf{r})$.

Neglect the electric potential produced by $\delta n(\mathbf{r})$ itself. Since the electric potential is weak, linearize $\delta n(\mathbf{r})$ in $\phi(\mathbf{r})$. Express the result in terms of the density of states at the Fermi energy, $D(E_F)$.

Hint: After the electrons have settled down, the "local Fermi energy" $\hbar^2 k_F^2(\mathbf{r})/2m + e\phi(\mathbf{r})$ must be *constant* in space, otherwise the electrons would keep redistributing themselves.

(d) Using the solution of Problem 5c, find a differential equation satisfied by the electric potential in a metal taking into account possible external charges as well as the local electron charge induced by the electric potential itself.

(e) An external charge q is placed in the electron gas described above. Using the solution of Problem 5d, find the self-consistent electric potential $\phi_q(\mathbf{r})$ produced by this charge.

What is the characteristic length Λ over which the electron density is perturbed and beyond which the electric potential essentially vanishes?

[*The length* Λ *is called the* screening length.]

(f) Find the numerical value of Λ for copper. Compare it with the average distance between the electrons.