HW#3 —Phys622—Spring 1997
Due Wednesday, Feb. 18, 1997
Turn in at or before beginning of class.

Prof. Ted Jacobson
Room 4117, (301)405-6020
jacobson@physics.umd.edu

1. Problem 2.1, Schwabl. (*c*) Show also that (*b*) yields $(2\pi)^3$ times the delta function $\delta^3(x)$ in the limit $\alpha \to 0$. [4 *points*]

2. Show that the expectation value of the momentum vanishes for a real wave function. [2 *points*]

3. Show that if the state $\psi(x)$ has mean momentum p, then the state $\tilde{\psi}(x) = e^{ip_0 x/\hbar}\psi(x)$ has mean momentum $p + p_0$. [2 *points*]

4. Show that the probability current density (eqn. (2.59) of Schwabl) for a wavefunction $\psi(x)$ is given by

$$\mathbf{j} = (\hbar/m)|\psi|^2\nabla\theta,$$

where θ is the phase of the wave function, $\psi = |\psi|e^{i\theta}$. [2 *points*]

5. Compute the position and momentum uncertainties Δx and Δp in the ground state of an infinitely deep one dimensional square well potential, and compare with the uncertainty relation $\Delta x \Delta p \geq \hbar/2$. [5 *points*]

Note: The uncertainty or *variance* ΔA of an observable in a state $|\psi\rangle$ is defined by

$$\begin{aligned} (\Delta A)^2 &\equiv \langle\psi|(A - \langle\psi|A|\psi\rangle)^2|\psi\rangle & (1) \\ &= \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2. & (2) \end{aligned}$$

6. A particle of mass m is in the ground state of an infinitely deep one-dimensional square well, when suddenly the width of the well increases from a to $2a$. Compute the probability that the particle will be found in the ground state of the larger well. If the width instead decreases from $2a$ to a what is this 'ground to ground' probability? [3 *points*]

7. For any Hamiltonian, show that the Heisenberg equations of motion preserve the canonical commutation relations, i.e., $[\hat{x}(t), \hat{p}(t)] = i\hbar$ for all t if it holds initially at $t = 0$. *Note*: The *Heisenberg picture* is explained in section 8.5 of Schwabl. [3 *points*]

8. Find the general solution ot the Heisenberg equation of motion for the position operator $\hat{x}(t)$ of a one-dimensional harmonic oscillator. Don't forget to require hermiticity of $\hat{x}$ and the canonical commutation relations $[\hat{x}, \hat{p}] = i\hbar$. [5 *points*]

9. Show that the Heisenberg lowering operator $a(t)$ for a harmonic oscillator evolves according to $a(t) = \exp(-i\omega t)\, a(0)$. *Hint*: Eqn (3.8) of Schwabl writes the harmonic oscillator Hamiltonian in terms of the lowering and raising operators a and $a^\dagger$, and (3.6) gives their commutation relations. [3 *points*]

10. FInd the (normalized) bound states and their energies for the one-dimensional delta function well $V(x) = -a\delta(x)$ $(a > 0)$ using the position representation. *Hint*: Is the derivative of the wavefunction continuous at $x = 0$? [7 *points*]

11. Express the stationary state Schrödinger equation in the momentum representation. *Hint*: The potential term should involve an integral over momenta, involving the Fourier transform of the potential. Section 8.3.2 of Schwabl may be helpful. [3 *points*]

12. Redo problem 10 using the momentum representation. [7 *points*]

13. This problem develops some properties of what are called "coherent states". Let operators a and $a^\dagger$ (= adjoint of a) satisfy $[a, a^\dagger] = 1$. We'll work here just with the algebra; do not specify a representation. Let $|0\rangle$ be a vector annihilated by a, i.e. $a|0\rangle = 0$. For each complex number z define a ket $|z\rangle$ by

$$|z\rangle = e^{za^\dagger - z^* a}|0\rangle. \tag{3}$$

Show that

(a) $a|z\rangle = z|z\rangle$

(b) $|z\rangle = e^{-|z|^2/2}\, e^{za^\dagger}|0\rangle$

(c) $\langle z|w\rangle = e^{z^* w - z^* z/2 - w^* w/2}$

(d) Find the average $\langle N\rangle = \langle z|N|z\rangle$, the uncertainty ΔN, and the ratio $\Delta N/\langle N\rangle$ for the *number operator* $N = a^\dagger a$ in the state $|z\rangle$. How does this ratio behave when $|z|$ is large?

Hint: The first three parts can be done using the operator identities shown in problem 9 of HW#1. [2+2+3+3=10 *points*]