HW#2 —Phys62—Spring 1997
Due Wednesday, Feb. 11, 1997
Turn in at or before beginning of class.

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1. Show that the adjoint of an operator $B^\dagger$ cf. HW#1.2) is defined by the property that, for all vectors v and w, [2 *points*]

$$\langle v|B^\dagger|w\rangle = \langle w|B|v\rangle^*. \tag{1}$$

2. The adjoint of a ket $|v\rangle$ is defined as the corresponding bra: $|v\rangle^\dagger = \langle v|$. Show that [2 *points*]

$$A|v\rangle^\dagger = \langle v|A^\dagger. \tag{2}$$

3. In the position representation, the ground state $|\psi_0\rangle$ of an electron in a hydrogen atom has the form:

$$\psi_0(\mathbf{r}) = \langle \mathbf{r}|\psi_0\rangle = \frac{e^{-r/a}}{\sqrt{\pi a^3}}, \tag{3}$$

where a is the Bohr radius, $\mathbf{r} = (x, y, z)$ is the radius-vector and $r = |\mathbf{r}|$ is its absolute value.

Calculate the wave function $\langle \mathbf{p}|\psi_0\rangle$ in the momentum representation. [5 *points*]

4. Tritium ^{3}H decays to ^{3}He via beta-decay (with a lifetime of 12.26y). What is the probability that an electron initially in the ground state of a tritium atom will wind up in the ground state of the ^{3}He-ion after the beta decay has occurred? (*Hint*: The time for the beta particle to leave the atom is small compared with the characteristic timescale for the electron in the ground state.) (*Useful information*: The ground state wavefunction of a hydrogen-like atom is given by (3), where $a = Z^{-1}a_B$ with a_B the Bohr radius and Ze the nuclear charge.) [5 *points*]

5. *Adapted from Physics Qualifier, Spring 1994, Problem II-5.*

The neutral B^0 meson and its antiparticle $\bar{B}^0$ are eigenstates of the strong interaction part of the Hamiltonian but not of the weak interaction part. Thus a B^0 can change into a $\bar{B}^0$. Here we consider a simplified version of this process. The problem involves a relativistic system, but it can be described without use of the Dirac equation.

Suppose that in the rest (zero momentum) frame of the meson, the Hamiltonian in the (orthonormal) basis $\{|B^0\rangle, |\bar{B}^0\rangle\}$ takes the form

$$H = \begin{pmatrix} \epsilon & \delta \\ \delta & \epsilon \end{pmatrix}$$

where δ and ϵ are constants. [3+7+2=12 *points*]

(a) Find the eigenvalues m_1, m_2 and corresponding "mass eigenstates" $|1\rangle$, $|2\rangle$ of H.

(b) Suppose a B^0 is created at time $t = 0$. Find the probability that after a time t in its rest frame, the B^0 would have converted into a $\bar{B}^0$. (The only thing you need to assume from relativistic quantum mechanics is that the phase of an energy eigenstate evolves by $\exp(-iEt/\hbar)$, where E is the *total* relativistic energy $E^2 = p^2 + m^2$ (in units with $c = 1$).

(c) What is the time required for a complete oscillation from B^0 into $\bar{B}^0$ and back to B^0? Also express your result in seconds with δ given in eV's.

6. Consider a Hamiltonian $H = \mathbf{p}^2/2m + V(\mathbf{x})$. Derive the "Thomas-Reiche-Kuhn sum rule"

$$\sum_n (E_n - E_a)|\langle n|x|a\rangle|^2 = \hbar^2/2m. \tag{4}$$

Here n labels a complete set of energy eigenstates, $|a\rangle$ is any fixed energy eigenstate, and x is the position operator for the x-coordinate. (*Hint*: Consider $\langle a|[[x, H], x]|a\rangle$, and insert a resolution of the identity $I = \sum_n |n\rangle\langle n|$ in a judicious location.) [7 *points*]

7. Consider representing the momentum operator in the position representation by

$$\tilde{p}_i = -i\hbar\partial_i + f_i(x)$$

where i is the spatial coordinate index. [2+5+3=10*points*]

(a) Under what conditions on $f_i(x)$ do $\tilde{p}_i$ and the usual x^i operator (multiplication by x^i) fulfill *all* the canonical commutation relations ($[x^i, x^j] = 0 = [\tilde{p}_i, \tilde{p}_j]$, $[x^i, \tilde{p}_j] = i\hbar\delta^i_j$)?

(b) For the acceptable $f_i(x)$'s, show that the representation is "unitarily equivalent" to the usual one (with $f_i = 0$). That is, find a change of orthonormal basis $|x\rangle \to |\tilde{x}\rangle$ for which $\langle x|\tilde{p}_i|x'\rangle = \langle\tilde{x}|p_i|\tilde{x}'\rangle$. Assume space is R^3.

(c) Suppose now that an infinite solid cylinder around the z-axis is removed, leaving behind a "hole" in space. Then the representations for different acceptable f_i's are not necessarily unitarily equivalent. Explain why and describe a physical measurement that could distinguish between the different representations.