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Read Sections 10.1,2 (addition of spins) and 13.2.1,3 (helium) of Schwabl.

1. *Physics Qualifier, January 1997, Problem II-4.*

(a) A beam of spin-1/2 particles moving along the y axis passes through two collinear Stern-Gerlach apparatuses, both with the lower (spin "down") beams blocked (the particles in the lower beam are absorbed). The first apparatus has its magnetic field along the z axis and the second has its magnetic field along the axis z' tilted to the angle ϑ relative to the z axis in the (x, z) plane.

What fraction of the particles leaving the first apparatus will exit the second one?

(b) If a third Stern-Gerlach apparatus that transmits only spin up along the z axis is introduced, what fraction of particles leaving the first apparatus will exit the third?

(c) If the middle filter transmits both spins up and down (no blocking), what fraction of particles leaving the first apparatus will exit the third if the up and down beams

i. remain separate, or
ii. are coherently recombined?

2. Suppose two spin-1/2 particles are in the singlet spin state. If the spin of one particle is measured and found to be 'up' in the direction $\hat{a}$, what is the probability that the other particle will be found to have spin 'up' in the direction $\hat{b}$?

3. In the hydrogen atom, apart from the Coulomb interaction, there exists another, the so-called *hyperfine* interaction between the magnetic moments of the electron and proton. The Hamiltonian of this interaction is:

$$\hat{H}_{hf} = A\,\hat{\mathbf{S}}_p \cdot \hat{\mathbf{S}}_e, \qquad A > 0, \tag{1}$$

where $\hat{\mathbf{S}}_p$ and $\hat{\mathbf{S}}_e$ are the spin operators of the proton and electron.

Is the total spin of the electron and proton a good quantum number for Hamiltonian (1)? Find the energy eigenstates and eigenvalues of Hamiltonian (1) and describe their quantum numbers and degeneracies. Describe how the hyperfine interaction (1) splits the ground state energy of the hydrogen atom.

The energy splitting ΔE of the ground state energy of the hydrogen atom caused by the hyperfine interaction (1) is equal to the energy of a photon of the wavelength $\lambda = 21.4$ cm: $\Delta E = hc/\lambda$. Microwave spectroscopy at this frequency is utilized to detect hydrogen in the Universe.

4. Two (not identical) particles of spin 1/2 are located at two different points separated by a distance $\mathbf{a} = a\,\mathbf{e}_z$ along the z axis. The particles interact via the dipole-dipole magnetic energy:

$$\hat{H} = \frac{\mu_1 \cdot \mu_2}{a^3} - 3\frac{(\mu_1 \cdot \mathbf{a})(\mu_2 \cdot \mathbf{a})}{a^5}, \tag{2}$$

where μ_1 and μ_2 are the operators of magnetic moments of the particles.

(a) Express the Hamiltonian (2) in terms of the spin operators of the particles and gyromagnetic factors γ_1 and γ_2, and find its eigenvalues and eigenvectors. Describe the quantum numbers of the eigenstates (i.e., of what operators are they eigenvectors?)

Hint: Use the fact that, for a spin-1/2 system, $S_z^2 = (\pm\hbar/2)^2 = (\hbar/2)^2$ to write the second term in terms of the z component of the *total* spin.

(b) Suppose a magnetic field $\mathbf{B} = B\,\mathbf{e}_z$ is applied to the system along the z axis. The Hamiltonian of the system acquires the additional Zeeman term:

$$\hat{H}' = -(\mu_1 + \mu_2) \cdot \mathbf{B}. \tag{3}$$

Find how the magnetic field modifies the energy eigenstates and eigenfunctions of the system. *Caution*: If $\gamma_1 \neq \gamma_2$, (3) does *not* just involve the z-component of the total spin. You will have to find the eigenvalues of a 2×2 matrix in order to solve this problem.

5. *Adapted from Qualifier, Fall 1979, II-1.*

Two spin-1/2 one-dimensional fermions interact via the following potential:

$$U(x_1 - x_2) = -\lambda\delta(x_1 - x_2)(2\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + 1), \tag{4}$$

where $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ are the spin operators of the fermions (and $\hbar = 1$).

Can the two fermions form a bound state if $\lambda > 0$? If so, write the energy, the wave function $\psi(s_1, x_1, s_2, x_2)$ (s_1 and s_2 are the z-axis projections of the fermions' spins), and the total spin of that state. Give the answers in two cases:

i. The fermions are distinguishable.

ii. The fermions are indistinguishable.

6. The two-body nuclear wave function of a diatomic molecule formed from two identical atoms has a symmetry that is tied to that of the nuclear spins.

(a) Which total angular momentum states are allowed for such a molecule and what are their degeneracies if

i. the nuclear spin is zero,

ii. the nuclear spin is 1/2 (as, for example, in a hydrogen molecule).

(b) Explain why transitions between rotational states of such molecules connect only states whose orbital angular momentum have the same parity.

(c) Suppose the temperature of a gas of hydrogen molecules is high enough so that many different angular momentum states are populated with essentially equal a priori probability. What will be the ratio of intensities of the rotational spectra for transitions between states of even or odd total angular momentum?

7. (a) What are the possible values for the degeneracy of the ground state and first excited state of a spinless particle in a bound state of a spherically symmetric potential?

   (b) Consider two noninteracting identical spin-1/2 particles in a spherically symmetric potential. What are the possible degeneracies for the ground state and the first excited state?

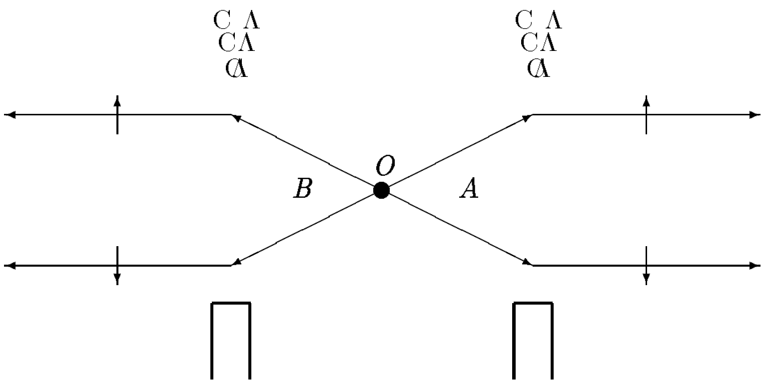
   (c) Consider two identical spin-1/2 particles in a spherically symmetric external potential and having a spin-spin interaction with Hamiltonian $H_{int} = -A\vec{S}_1 \cdot \vec{S}_2$ where $A > 0$. What are the possible ground state degeneracies if

      i. $A\hbar^2 > E_1 - E_0$

      ii. $A\hbar^2 < E_1 - E_0$

   where E_0 and E_1 are the ground and first excited energies for a single particle in the potential? For each possible value of the degeneracy, list a complete set of the corresponding state vectors. Use the notation $|nlm\rangle_{1,2}$ for the orbital states of particles 1 and 2 and $|\pm\rangle_{1,2}$ for the up and down spin states in the $\hat{z}$ direction.

8. A spin-zero particle O decays into two spin-1/2 particles A and B, which fly to the right and to the left, respectively, through Stern-Gerlach apparatuses, which allow the spin projections of the particles on the z axis to be measured, as shown in the Figure below.



The wave function of the system after decay of the O particle is

$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \qquad (5)$$

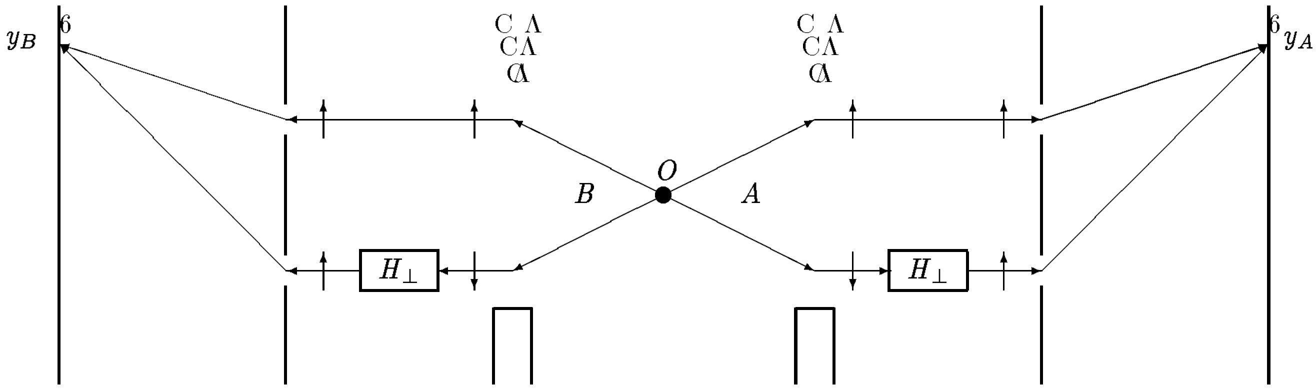
where the right-hand arrow indicates the spin projection of particle A and the left-hand one the spin projection of particle B.

In the following you are asked for the probabilities for different things to happen. Most or all of these can be reasoned without a heavy formal approach, but it is often difficult to explain such reasoning. Therefore a formal argument may be preferred. It would be best if you argue both ways! To make formal arguments, remember that the probability of observing a system in a quantum state $|\psi\rangle$ to have a property λ is given by the expectation value $\langle\psi|P_\lambda|\psi\rangle$ of the projection operator P_λ onto the subspace of states having the property λ. For example, in the problems below, λ might be the property of particle A having spin up, or of falling at the point y_A.

(a) Suppose the spin projection of particle A is measured, while no measurement is done on particle B. What are the probabilities to find particle A in the spin-up and in the spin-down states?

(b) Suppose the spin projection of particle B has been measured first and found to be up. Then, the spin projection of particle A is measured. What are the probabilities to find particle A in the spin-up and in the spin-down states in this case?

Now, as in Problem 7b of Homework 11, magnetic fields are applied perpendicular to the z axis in the spin-down paths. The strength of the fields is such that the spins of the particles rotate from the down to the up states during the flight time. Then, the beams pass through slits, and we measure the probability $P(y_A, y_B)$ that particle A arrives at a point y_A and particle B at a point y_B within a short period of time ("simultaneously"). The latter condition assures us that both particles originated from the same decay event, not from decay of different O particles.



(c) Let us consider only those events where particle B arrives at a certain point y_B. For those events, does the probability to find particle A at an arbitrary point y_A exhibit an interference pattern? Explain.

(d) Suppose we don't care where the particle B arrives and measure only the probability distribution of the particle A: $P(y_A) = \int dy_B\, P(y_A, y_B)$. Does this function exhibit an interference pattern? Explain.

(e) Assuming that the distance from the slits to the screens is big compared with both the distance between the slits and the characteristic y's, show that $P(y_A, y_B)$ is a function of only the relative distance $y_A - y_B$.

The state (5) is a typical example of a so-called entangled state, a state of several particles whose wave function is not a product of wave functions of individual particles: $\psi(A, B) \neq \psi(A)\psi(B)$.