**HW#11** —Phys622—Spring 1998 | Prof. Ted Jacobson
Due Friday, May 1, 1998 | Room 4117, (301)405-6020
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*Read chapter 9 of Schwabl.*

**1.** Consider a charged particle in an arbitrary electromagnetic field. The Hamiltonian of the system is:

$$H = \frac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m} + e\phi, \tag{1}$$

where $\phi(\mathbf{x}, t)$ is the scalar potential and $\mathbf{A}(\mathbf{x}, t)$ is the vector potential. In class we showed that the time derivative of the position operator in the Heisenberg picture is $\dot{x}^i = (p^i - \frac{e}{c}A^i)/m \equiv v^i$. Thus the Hamiltonian can be written as $H = \frac{1}{2}mv^2 + e\phi$.

**(a)** Show that $[v^i, v^j] = (i\hbar e/m^2 c)\epsilon^{ijk}B^k$.

**(b)** Use part 1a to find the time derivative of the velocity operator and compare with the classical Lorentz force law:

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{2}$$

*Note:* Don't forget that $v^i$ may have some explicit time-dependence.

**2.** It is usually said that one needs the Dirac equation to derive the gyromagnetic ratio for the electron. Feynman pointed out another way to reach the same conclusion: to incorporate spin into the nonrelativistic Schrödinger equation, replace $(\mathbf{p} - e\mathbf{A}/c)^2$ by $(\sigma\cdot(\mathbf{p} - e\mathbf{A}/c))^2$ in the hamiltonian (1). Show that this leads to the Pauli Hamiltonian (9.40) in Schwabl. *Hint*: Use (9.18).

**3.** A neutron travels along the positive $z$ axis with velocity 2000 m/s. Initially, it travels in a region (region 1) of uniform magnetic field $B_0\hat{z}$ with spin 'up' in the $\hat{z}$ direction. It then enters another region (region 2), where the magnetic field is $\vec{B} = B_0\hat{z} + B_1\hat{x}$. After travelling for 20 m in this region without any deflection the neutron enters a third region where the magnetic field is once again $B_0\hat{z}$. Suppose $B_1 = 0.01B_0$, and $\mu_n B_0/\hbar = 60\mathrm{s}^{-1}$, where $\mu_n$ is the magnetic moment of the neutron. Calculate the probability that the spin is 'down' in the $\hat{z}$ direction after the neutron has traveled through region 2.

**4.** **(a)** Show that, for any spinor $|\alpha\rangle$, there is a unit vector $\mathbf{n}$ such that $\mathbf{n}\cdot\sigma|\alpha\rangle = |\alpha\rangle$. That is, $|\alpha\rangle$ is the spin "up" state in the direction $\mathbf{n}$.
Here is a nice way to proceed: Let $|\alpha\rangle$ and $|\beta\rangle$ be an orthonormal pair of spinors. Define $N = |\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|$. Show that $N|\alpha\rangle = |\alpha\rangle$, and that $N$ is hermitian, $Tr\,N = 0$, and $N^2 = 1$. Argue that these properties imply that $N = \mathbf{n}\cdot\sigma$ for some $\mathbf{n}$.

(b) Show that the orthogonal spinor $|\beta\rangle$ referred to in part 4a corresponds to the spin "down" state in the direction $\mathbf{n}$, i.e. $\mathbf{n}\cdot\sigma|\beta\rangle = -|\beta\rangle$.

**5.** Let us study the transformation of spin-1/2 wave functions under time reversal.

(a) Under time reversal, angular momentum changes sign, so spin up should become spin down and vise versa. That must be true for any choice of spin quantization axis.

Show that the operator

$$\hat{T} = -i\sigma_y\hat{K} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\hat{K}, \tag{3}$$

where $\hat{K}$ is the operator of complex conjugation, is a time reversal operator, by showing that applying it to any spinor $|\alpha\rangle$ yields the (unique up to phase) orthogonal spinor. (Problem 4b shows that the orthogonal spinor has the opposite spin direction.)

The complex conjugation operator $\hat{K}$ is an *antiunitary* operator, which has the following property:

$$\hat{K}(\alpha|a\rangle + \beta|b\rangle) = \alpha^*\hat{K}|a\rangle + \beta^*\hat{K}|b\rangle, \tag{4}$$

which differs from the similar property of a linear operator $\hat{L}$:

$$\hat{L}(\alpha|a\rangle + \beta|b\rangle) = \alpha\hat{L}|a\rangle + \beta\hat{L}|b\rangle, \tag{5}$$

(b) Using (3), show that

$$\hat{T}^2 = -1. \tag{6}$$

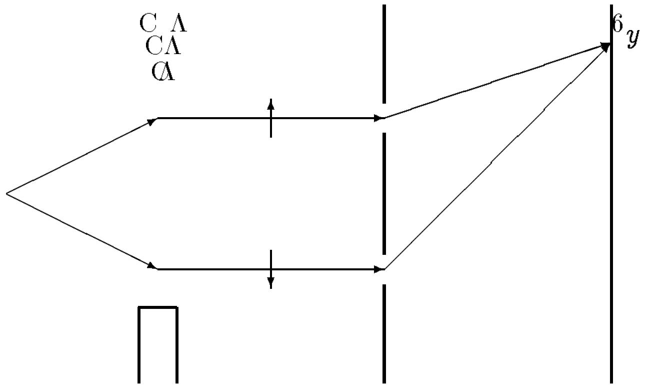
(c) Suppose that a Hamiltonian $\hat{H}$ of a spin-1/2 system has time-reversal symmetry, so that $\hat{H}$ and $\hat{T}$ commute. Show that each energy eigenstate is at least doubly degenerate. (*Hint*: Can $\hat{T}$ have an eigenvector?)

*This is the so-called Kramers degeneracy. Because any higher spin can be represented as a composition of spins 1/2, statements 5b and 5c are valid for any half-integer spin.*

**6.** Problem 9.3 of Schwabl [*rotation of a spinor*]. Do this two ways, (i) the way Schwabl suggests in his hint, and (ii) by expanding the exponential operator $\exp(-i\alpha S_y/\hbar)$.

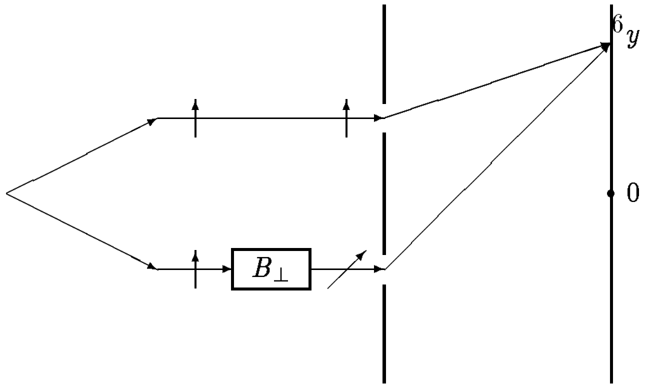
**7.** When spinless particles pass through two slits, they exhibit an interference pattern (see Fig. 2.1c). Let us discuss how this picture is modified when the particles have spin 1/2.

(a) [**3 points**] A beam of spin-1/2 particles with some generic polarization (not along the $z$ axis) passes through a Stern-Gerlach apparatus (shown in the Figure below as the wedge and the rectangle). Then, the spin-up and spin-down beams pass through separate slits, and the probability $P(y)$ to register the particles is measured as a function of the coordinate $y$.




Does $P(y)$ exhibit an interference pattern for the spin-1/2 particles in the setup described above? (Yes, No)

**(b)** A beam of spin-1/2 particles polarized up along the $z$ axis is split into two parts. One part passes through a region with a magnetic field $B_\perp$ perpendicular to the $z$ axis, as shown in the Figure below by the rectangle with $B_\perp$.




During the flight through the region with the magnetic field, the polarization of the particles rotates by an angle $\theta$, which depends on the strength of the magnetic field. By varying magnetic field, we can change the rotation angle $\theta$.

i. [**3 points**] Describe qualitatively how the amplitude of the interference pattern depends on $\theta$.

ii. [**3 points**] In zero magnetic field ($\theta = 0$), the interference pattern has the maximum at the point $y = 0$ between the two slits.
When $\theta = 2\pi$, does the interference pattern have a maximum at the point $y = 0$? What is the answer when $\theta = 4\pi$?