HW#10 —Phys622—Spring 1998
Due Friday, April 24, 1998
Turn in at or before beginning of class.

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Read chapter 7, section 8.6, and appendix B of Schwabl.

1. *Adapted from Physics Qualifier, January 1983, Problem II-1.*

A particle of mass m experiences the following three-dimensional potential:

$$V(r) = -\frac{\lambda_1}{r} + \frac{\lambda_2}{r^2}, \tag{1}$$

where $\lambda_1, \lambda_2 > 0$.

Find the discrete energy spectrum and compare its degeneracy with that of the hydrogen atom. Write the wave functions of the two lowest energy levels for a given angular momentum l (do not bother to normalize). *Hint*: Use the results for the Coulomb problem.

2. In problem HW#6.4 you studied a particle of mass m and charge e in a constant electric field, using a time-independent scalar potential $\phi(x) = -Ex$ for the electric field. You found the energy eigenstates in the momentum representation, and showed that the probability current vanishes for these eigenstates, and interpreted these eigenstates in terms of a superposition of "forward and backward" motions of the charge.

The same electric field can be obtained from a *space*-independent but time-dependent vector potential. In this gauge the canonical momentum in the x direction commutes with the hamiltonian, so is conserved. You can easily solve the time-dependent Schrödinger equation using the position representation, assuming a momentum eigenstate for all three components of momentum.

(a) Write the hamiltonian in a space-independent gauge.

(b) Solve the time-dependent Schrödinger equation for momentum eigenstates. Use periodic boundary conditions to normalize your solution.

(c) Compute the probability current density (see problem 7.1 in Schwabl) for your solution of part 2b, and interpret the result in terms of the classical orbits.

3. *Adapted from Physics Qualifier, January 1991, Problem II-5.*

Let us consider a two-dimensional particle in the homogeneous magnetic field in the so-called *Landau gauge*:

$$A_y = Bx, \qquad A_x = 0. \tag{2}$$

(a) By writing explicitly the Schrödinger equation in the position representation and exploiting the analogy with an oscillator, find the energy eigenvalues and eigenfunctions of the Hamiltonian (7.2, *Schwabl*) with the vector potential (2). (*Hint*: Use the fact the $[p_y, H] = 0$.) Is there any degeneracy of the eigenstates? Sketch schematically an eigenfunction $\psi^{(0)}(x, y)$ of the lowest energy level. What can you say about the localization of the wave function in the x and y directions?

(b) You should have found in Problem 3a that the energy eigenfunctions are degenerate in p_y, the eigenvalue of the momentum in the y direction. Any superposition of degenerate eigenstates is also an eigenstate. Thus, we may transform a set of degenerate eigenstates $\psi_{p_y}^{(n)}(x, y)$, where n is the number of the Landau level, into another set of degenerate eigenstates labeled by an index Y:

$$\psi_Y^{(n)}(x, y) = \int e^{-ip_yY/\hbar}\psi_{p_y}^{(n)}(x, y)dp_y. \tag{3}$$

Perform transformation (3) explicitly for the lowest Landau level ($n = 0$). What can you say about the localization of a wave function $\psi_Y^{(0)}(x, y)$ in the x and y directions?

(c) Evaluate the probability current density

$$\mathbf{j} = \frac{\hbar}{2mi}\Big(\psi^*\nabla\psi - (\nabla\psi^*)\psi\Big) - \frac{e}{mc}\mathbf{A}\psi^*\psi, \tag{4}$$

for a lowest level eigenfunction $\psi_{p_y}^{(0)}(x, y)$.

Explain how your result is consistent with the classical picture of circular electron orbits.

(d) Suppose the particle's motion is restricted by the width L_y in the y-direction and length L_x in the x-direction, with both sizes much greater than the natural characteristic width of a wave function.

Calculate the degeneracy of each energy eigenvalue. To simplify the problem assume that a periodic boundary condition is imposed in the y direction:

$$\psi(x, y) = \psi(x, y + L_y). \tag{5}$$

Condition (5) allows only discrete values of the index p_y that labels the eigenstates found in Problem 3a. Use this discreteness to count the degeneracy, i.e. how many orthogonal states are there at each Landau level?

(e) Suppose each eigenstate can be occupied by only one electron. (The electrons are fermions, and let us neglect the spin for simplicity.) Suppose the electrons completely fill all of the states of the N lowest Landau levels.

Using the solution of Problem 3d, calculate the number of electrons per unit area of the (x, y) space. Calculate the magnetic flux per electron and compare the result with the *quantum of magnetic flux* (cf. (7.27) of Schwabl):

$$\Phi_0 = \frac{\pi\hbar c}{e}. \tag{6}$$

4. *Adapted from Physics Qualifier, January 1991, Problem II-5.*

Now let us consider Problem 3 in the presence of a constant electric field E_x along the x direction.

(a) By writing explicitly the Schrödinger equation in the position representation and exploiting the analogy with an oscillator, find the energy eigenvalues and eigenfunctions of Hamiltonian (7.2, *Schwabl*) with the vector potential (2) and scalar potential $\phi = -E_x x$. Compare the eigenstates with those found in Problem 3a. Explain why the degeneracy is lifted.

(b) Calculate the electric currents J_x ,J_y across a surface of constant x and y respectively for an eigenstate found in Problem 4a.

What is the orientation of the current relative to the direction of the electric field? Compare your result with the classical result.

[*The occurrence of electric current in the direction perpendicular to both magnetic and electric fields is called the* Hall effect. *It was discovered by E. H. Hall in 1879 at the Johns Hopkins University in Baltimore.*]

(c) Suppose the two-dimensional system is filled with electrons at the concentration of n electrons per unit area. Calculate the Hall resistance, defined as the ratio of the voltage V_x in the x direction to the total current J_y in the y direction:

$$R_H = \frac{V_x}{J_y}. \tag{7}$$

Suppose that the electron concentration is such that N Landau levels are completely filled (see Problem 3e). Calculate the Hall resistance in this case and compare the result with the *quantum of resistance*:

$$R_0 = \frac{h}{e^2}. \tag{8}$$

Calculate the numerical value of R_0 in ohms.

5. In a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, using the symmetric gauge $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$, a generic wave function of the lowest Landau level has the form:

$$\psi_0(x,y) = f(z) \exp\left(-\frac{eB}{4\hbar c}|z|^2\right), \tag{9}$$

where $f(z)$ is an *analytic function* of the variable $z = x + iy$, that is a function that contains any powers of z in its Taylor expansion, but no powers of z^*:

$$f(z) = \sum_{m=0}^{\infty} C_m z^m \tag{10}$$

(see Schwabl, eqn. (8.93)).

(a) As shown on p. 178 of Schwabl, the probability density of the wave function (9) with $f(z) = z^m$ is concentrated at the radius $r_m = \sqrt{2m+1} r_0$, where $r_0 = \sqrt{\hbar c/eB}$ is called the *magnetic length*. Describe the current density in this state and interpret this state in terms of the classical electron orbits.

(b) Suppose the system has the shape of a disk of a large radius R. In this case, the largest possible value of m is restricted roughly by the condition that $r_m < R$. Calculate the number M_R of lowest Landau level states that fit into the disk. Show that the number of these states per unit area is the same as the number found in Problem 3e of this homework (for just the lowest Landau level).

(c) Suppose that $M = M_R$ electrons (fermions) occupy all available states of the lowest Landau level that were introduced in Problem 5b. (Suppose the spins are all aligned, and therefore neglect the spin.) Then the ground state is the Slater determinant (see, e.g., eqn (13.12) of Schwabl) formed with these M states. Show that the wave function of this state has the form:

$$\psi(z_1, z_2, \ldots, z_M) = \exp\left(-\frac{1}{4r_0^2}\sum_{i=1}^{M}|z_i|^2\right)\prod_{i=1}^{M}\prod_{j=1}^{i-1}(z_i - z_j). \tag{11}$$

To prove this statement, it suffices to show that

i. (11) contains only the wave functions of the lowest Landau level;

ii. all available lowest Landau level states are used in (11);

iii. (11) changes sign when any two electrons are exchanged.

(d) Now suppose that the number of electrons M is less than the number M_R of lowest Landau level states. Explain in words why (11) is no longer the correct ground state.

In the case where the *filling factor* $\nu = M/M_R$ is the inverse of an odd integer, Laughlin proposed the wave function which modifies (11) by the replacement $(z_i - z_j) \to (z_i - z_j)^{1/\nu}$. Since $1/\nu$ is odd, Laughlin's wave function is antisymmetric, and the largest power of z_i that appears is $M/\nu = M_R$, so the wavefunction just fits inside the sample. Also the wavefunction vanishes rapidly as z_i approaches z_j, thus incorporating the electron-electron repulsion. Apparently this Laughlin wavefunction is a good approximation for not too small filling factors.