HW#1 —Phys62—Spring 1997
Due Wednesday, Feb. 4, 1997
Turn in at or before beginning of class.

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Sections in Schwabl related to this homework set: 2.4.3, 2.8.2, 8.1, 8.2.

1. Find the trace, determinant, eigenvectors and eigenvalues of the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Which of these matrices is hermitian? [10 *points*]

2. The *hermitian conjugate* or *adjoint* $B^\dagger$ of a linear operator B is defined by the requirement that $(v, Bw) = (B^\dagger v, w)$ for all vectors v and w, where $(\ ,\ )$ is the inner product on the vector space. (We ignore here infinite dimensional subtleties.) Show from this definition that [6 *points*]

(a) $(B^\dagger)^\dagger = B$

(b) $(BC)^\dagger = C^\dagger B^\dagger$

(c) $(B^\dagger)_{ik} = (B_{ki})^*$, where $B_{ik} \equiv (e_i, Be_k)$ is the matrix element of B in the orthonormal basis $\{e_i\}$.

3. If A and B are hermitian operators show that although nothing can be said in general about AB, both $\{A, B\} \equiv (AB + BA)$ and $i[A, B] \equiv i(AB - BA)$ are hermitian. [2 *points*]

4. Show that a product of unitary operators is unitary. [2 *points*]

5. Show that the eigenvalues of a hermitian operator are real, and those of a unitary operator have unit modulus. [4 *points*]

6. A *projection operator* P is a hermitian operator that satisfies $P^2 = P$. (a) What are the possible eigenvalues of P, and (b) what is the diagonal form of P? [4 *points*]

7. Show that $\mathrm{Tr}\ |a\rangle\langle b| = \langle b|a\rangle$. [3 *points*]

8. The *commutator* of two operators is defined by $[A, B] = AB - BA$. Establish the following identities: [1+1+2+2=6 *points*] (**a**) $[A,B]=-[B,A]$

(**b**) $[A,B+C]=[A,B]+[A,C]$

(**c**) $[A,BC]=[A,B]C+B[A,C]$

(**d**) $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$ *Jacobi identity*

9. Let A and B now be two operators that commute with their commutator, i.e.

$$[A,[A,B]]=0=[B,[A,B]].$$

Show that

(**a**) $[A^n,B]=nA^{n-1}[A,B]$

(**b**) $[e^A,B]=e^A[A,B]$

(**c**) $e^ABe^{-A}=B+[A,B]$

(**d**) $e^Ae^B=e^Be^Ae^{[A,B]}$

(**e**) $e^Ae^B=e^{A+B}e^{[A,B]/2}$

Hints: Use (**a**) to show (**b**), (**b**) to show (**c**), and exponentiate (**c**) to show (**d**). To prove (**e**), show that $f(s)\equiv e^{sA}e^{sB}$ satisfies $df/ds=(A+B+s[A,B])f$, and integrate. [10 *points*]

10. Let A and B now be two *arbitrary* operators. Although 9c does not in general hold, the *Campbell-Baker-Hausdorff* identity is true:

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\cdots \quad (1)$$

Prove this identity by showing that $g(s)\equiv e^{sA}Be^{-sA}$ satisfies $dg/ds=[A,g]$ and integrating this equation. *Hint*: Think of $[A,\ ]$ as an operator on operators...at least this is the way I think of it. [3 *points*]

11. Consider representing the momentum operator in the position representation by

$$\tilde{p}_i=-i\hbar\partial_i+f_i(x)$$

where i is the spatial coordinate index. [2+5+3=10*points*]

(**a**) Under what conditions on $f_i(x)$ do $\tilde{p}_i$ and the usual x^i operator (multiplication by x^i) fulfill *all* the canonical commutation relations ($[x^i,x^j]=0=[\tilde{p}_i,\tilde{p}_j]$, $[x^i,\tilde{p}_j]=i\hbar\delta^i_j$)?

(**b**) For the acceptable $f_i(x)$'s, show that the representation is "unitarily equivalent" to the usual one (with $f_i=0$). That is, find a change of orthonormal basis $|x\rangle\to|\tilde{x}\rangle$ for which $\langle x|\tilde{p}_i|x'\rangle=\langle\tilde{x}|p_i|\tilde{x}'\rangle$. Assume space is R^3.

(**c**) Suppose now that an infinite solid cylinder around the z-axis is removed, leaving behind a "hole" in space. Then the representations for different acceptable f_i's are not necessarily unitarily equivalent. Explain why and describe a physical measurement that could distinguish between the different representations.