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S9.3 Motion in the Schwarzschild spacetime

Consider the action for a test particle in a gravitational field, $S = -mc^2 \int ds$, where the field is that of a central mass. The line element in this field is

$$ds^2 = F(r)\, dt^2 - \frac{1}{F(r)} dr^2/c^2 - (r^2/c^2)(d\theta^2 + sin^2\theta d\phi^2), \tag{1}$$

where $F(r) = 1 - r_g/r$, and r_g is the *Schwarzschild radius*, $r_g = 2GM/c^2$. This is the unique spherically symmetric vacuum solution to the Einstein gravitational field equations, up to coordinate changes. The parameter M is the mass of the central body. [If there is no central body, as would happen if a star collapsed to form a black hole, then this line element applies all the way down to the radius $r = r_g$, where $F = 0$. This is the event horizon of the black hole. The blow up of the line element at $r = r_g$ does not signify any physical divergence. Rather, this (Schwarzschild) coordinate system is not well-behaved at $r = r_g$. For this problem we restrict to the region outside the horizon.]

(a) The nonrelativistic limit corresponds to the conditions $r \gg r_S$, $dr/dt \ll c$, and $d\phi/dt \ll c$. Show that in this limit, the action is approximately equal to the nonrelativistic action, $S_{NR} = \int (\frac{1}{2}mv^2 + GMm/r)dt$, minus a constant rest mass term.

(b) Now let λ be an initially arbitrary parameter for the spacetime path, $t(\lambda), r(\lambda), \phi(\lambda), \theta(\lambda)$ and write the relativistic action in the form $\int L_\lambda \, d\lambda$. Restricting to motion in the equatorial plane, show that in units with $c = 1$ and with $\dot{} = d/d\lambda$ we have

$$L_\lambda = -m(F\dot{t}^2 - F^{-1}\dot{r}^2 - r^2\dot{\phi}^2)^{1/2}. \tag{2}$$

(c) L_λ is independent of t and ϕ, reflecting the time translation and rotational symmetries of the field, and there are corresponding conserved quantities, $p_t = \partial L_\lambda/\partial \dot{t}$ and $p_\phi = \partial L_\lambda/\partial \dot{\phi}$. Find the expressions for p_t and p_ϕ.

(d) Show that in the nonrelativistic limit $p_t \approx -(m - GMm/r + \frac{1}{2}mv^2)$, i.e. p_t is approximately minus the sum of the rest energy and the nonrelativistic mechanical energy.

(e) Suppose now that λ is not an arbitrary parameter, but is in fact the proper time along the worldline of the particle, so that $L_\lambda = -m$. Use the conservation laws to express this relation in terms of $\dot{r}$, r, and the conserved quantitites p_t and p_ϕ. Write the result in the form

$$\tfrac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) = \mathcal{E}, \tag{3}$$

where the effective potential depends on p_ϕ and the "energy" $\mathcal{E}$ depends on p_t. This is a conservation law and can be used to find the orbits. Show that this effective potential agrees with the one in problem S5.2. [The systems are not the same, since the dot here represents $d/d\tau$, not d/dt. However, in the nonrelativistic limit the distinction between d/dt and $d/d\tau$ can be neglected.]

S9.4 Motion in a gravitational wave

The line element describing a plane gravitational wave of frequency ω propagating in the x direction can be written (in $c = 1$ units) as

$$ds^2 = dt^2 - dx^2 - \big(1 + h\sin[\omega(t-x)]\big)dy^2 - \big(1 - h\sin[\omega(t-x)]\big)dz^2. \tag{4}$$

(a) Show that if a test mass governed by the action $-m\int ds$ is initially "at rest" in these coordinates (i.e. if $dx/dt = dy/dt = dz/dt = 0$ initially), then it remains so for all times.

(b) Show that if a ring of independent test masses in the $x = 0$ plane is at rest with respect to these coordinates at $t = 0$, then the physical shape of the ring, defined by the invariant distance between points on the ring at a constant value of the t coordinate, oscillates between two perpendicular ellipses.